AUGUR

Uncovering the Hidden Triangles of Uncertainty Analysis

ICEAA Professional Development & Training Workshop

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Speaker Bios

Julia Peters – Analyst

- 1 year of experience in DoD cost estimation
- BS in Physics – George Washington University
- Background experience in STEM education
- Research on laser calibration and photogrammetry

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- 4+ years of industry experience
- MA in Applied Economics – George Washington University
- Background in schedule and performance management
- Supports Navy Directed Energy programs

Outline

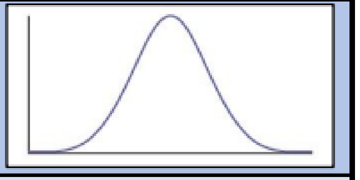
- Introduction
- Vectors
- Independent Uncertainties
- Correlated Uncertainties
- Application in ACE
- Summary
- Q&A

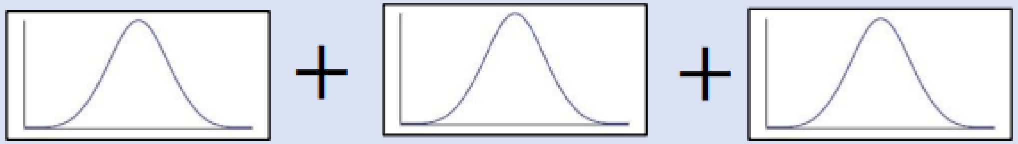
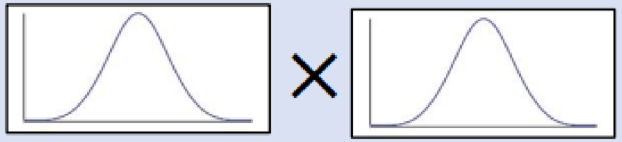
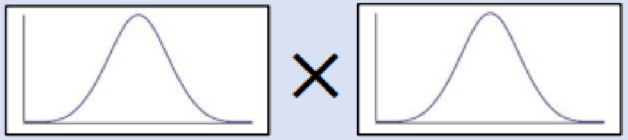
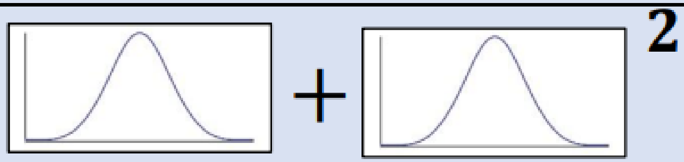
Introduction

Problem: Uncertainty analysis is crucial in the cost estimating process, yet the tools for quantifying uncertainty are mathematically abstract.

Goal: Provide estimators with a conceptual framework for uncertainty analysis that is grounded in geometry.

Top-Down vs Bottom-Up Applications

Top - Level Application of Cost Uncertainty		
WBS Level	**WBS Element**	**Application of Risk**
1	**Total Contract Cost**	
2	Management	*None*
2	Development Labor	*None*
2	Prototype Materials	*None*
2	Equipment	*None*

Bottom - Level Application of Cost Uncertainty		
WBS Level	**WBS Element**	**Application of Risk**
1	**Total Contract Cost**	*Composition of Children*
2	Management	
2	Development Labor	
2	Prototype Materials	
2	Equipment	

(Koellner, 2023)

Independent and Correlated Standard Deviations

Independent Case

$$\sigma_{total} = \sqrt{\sum_{i=1}^{n} \sigma_i^2}$$

Correlated Case

$$Var(Cost) = \sum_{i=1}^{n} \sigma_i^2 + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \rho_{x_i, x_j} \sigma_i \sigma_j$$

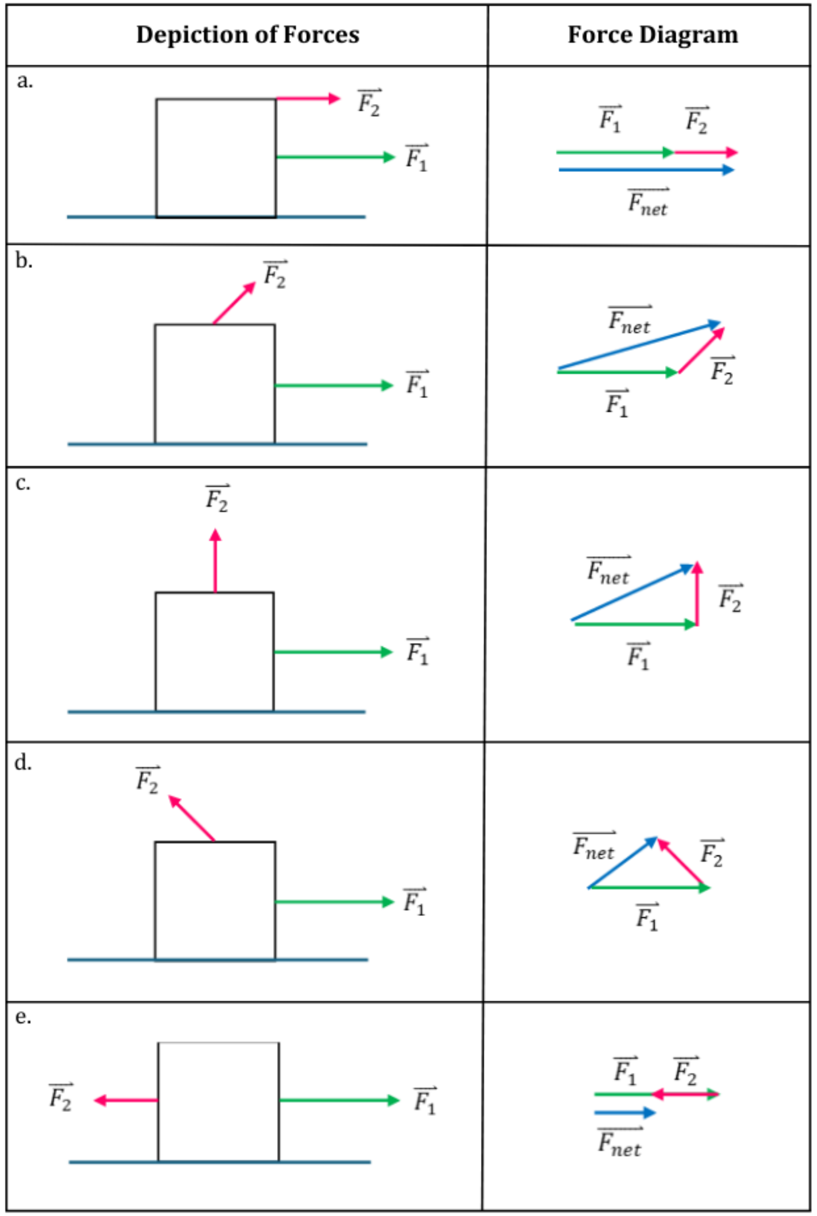
For each case:

- Take n=2 to examine the formula in 2 dimensions
- Map the formula onto its geometric analog
- Analyze triangles to visualize the relationship between correlation and cumulative uncertainty

Part 2: Vectors

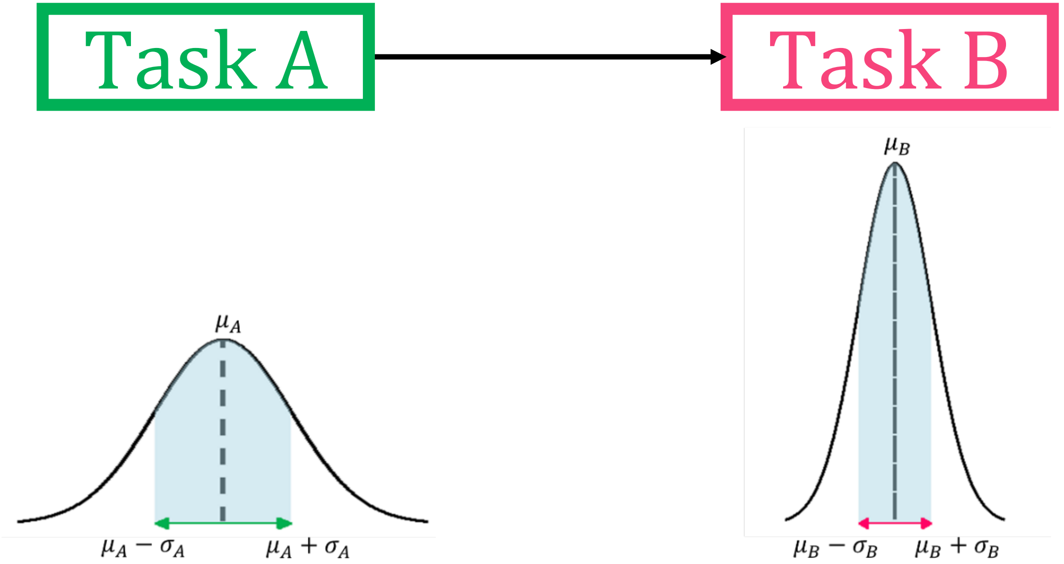
Vector Definition & Application

- **Scalar:** a quantity fully specified by magnitude
- **Vector:** a quantity with magnitude *and* direction
- Consider a box on a surface that is subject to two external forces. Which way will it move?
  - The box's motion will depend on the direction of the force vectors



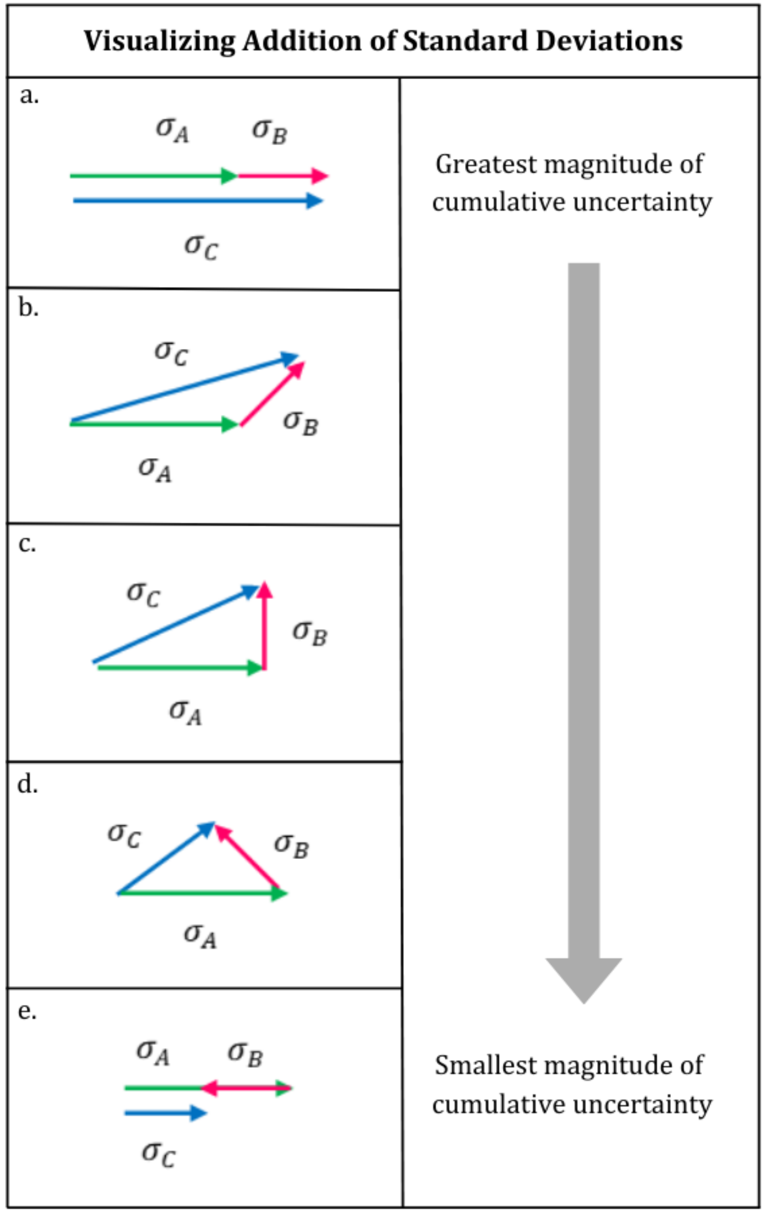
Standard Deviations as Vectors

- Assume Task A precedes Task B and each task has its own uncertainty
- Mean Completion Duration = Mean(A) + Mean(B)
  - The mean duration only considers magnitude
- If Task A increases in duration, is Task B likely to increase or decrease in duration as well?
  - The variance considers directional relationships



Application of Standard Deviation

- The magnitude of the cumulative standard deviation = σ_C
- Directional orientation of the of the child-level standard deviations = σ_A and σ_B
- σ_C depends on σ_A and σ_B
- The angle between σ_A *and* σ_B serves the same purpose as the correlation coefficient
  - Quantifies the alignment of the vectors relative to one another



Part 3: Independent Uncertainties and the Pythagorean Theorem

Independent Random Variables

$$\sigma_{total} = \sqrt{\sum_{i=1}^{n} \sigma_i^2}$$

Summation of independent standard deviations

$$\sigma_{total} = \sqrt{\sigma_1^2 + \sigma_2^2}$$

Reduce to 2D case

$$a^2 + b^2 = c^2$$

Pythagorean Theorem

Adding Independent Uncertainties

Let's represent our two uncertainties as **lengths** in the 2D plane.

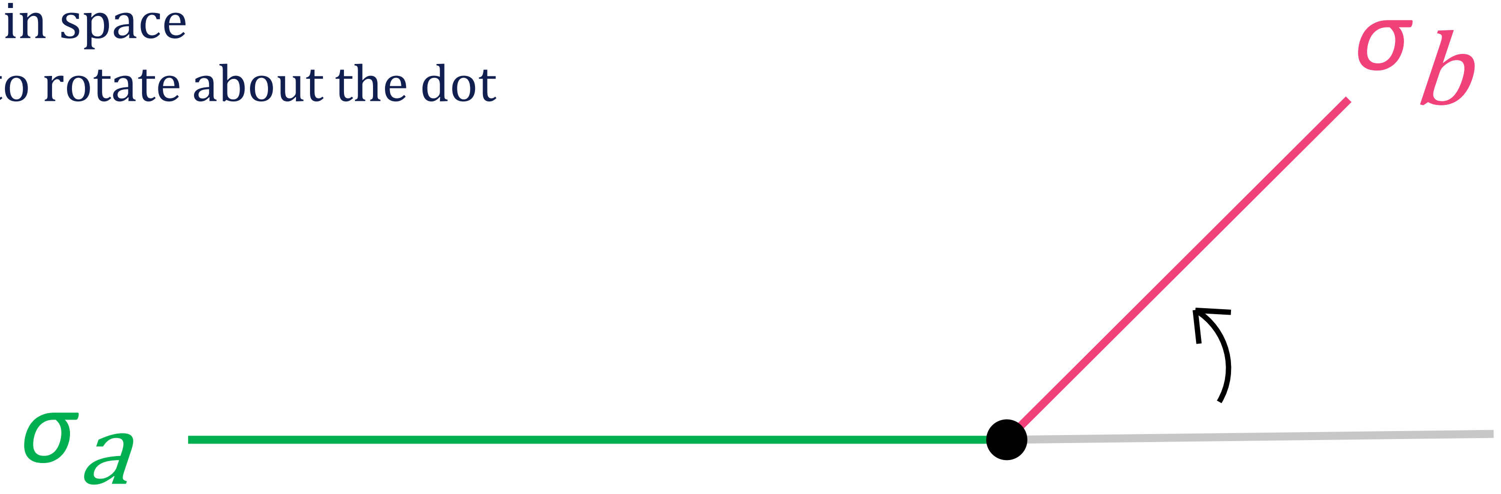
σ_a

σ_b

Distribution A has a wider spread, so its uncertainty vector σ_a is longer.

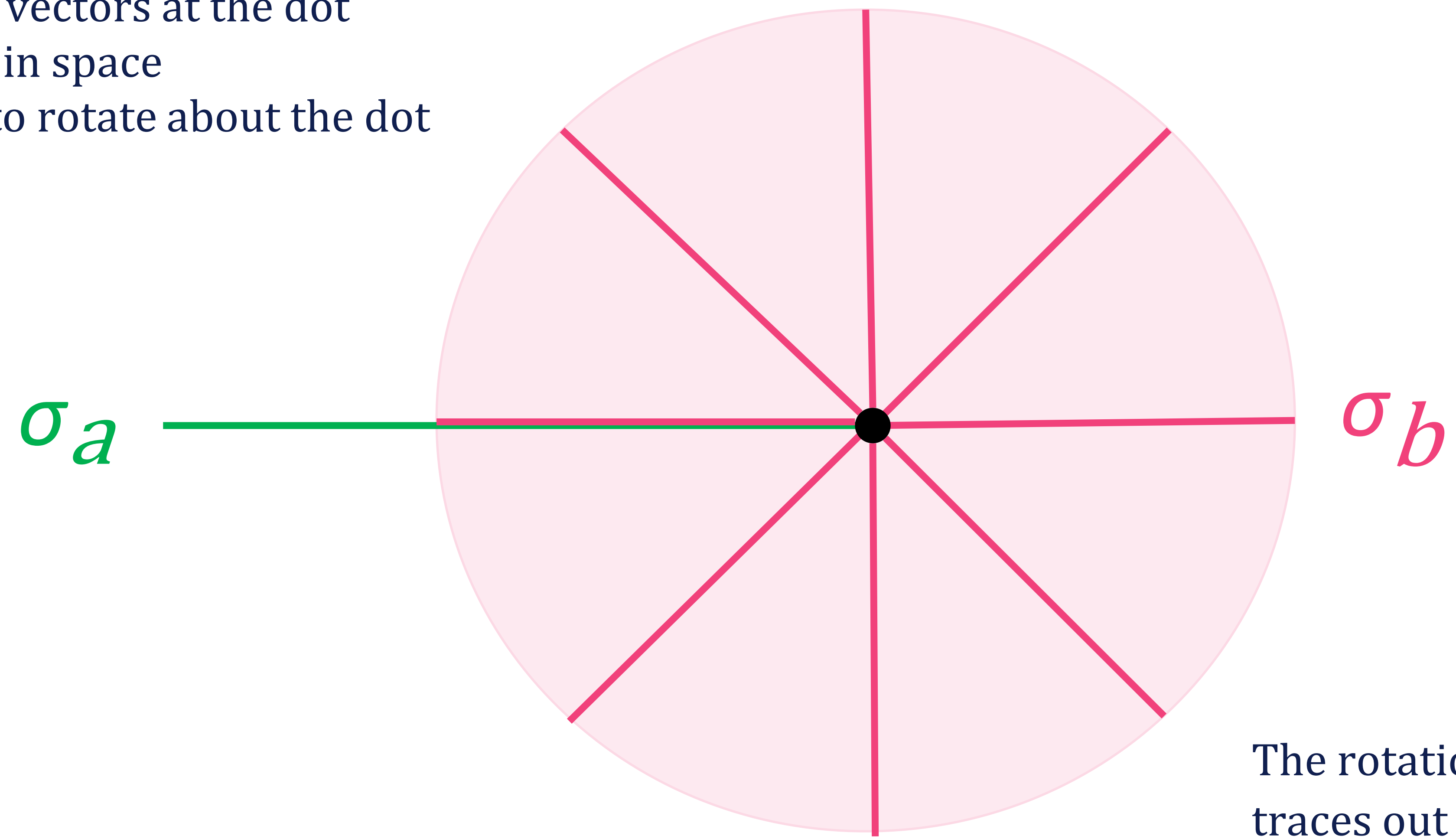
Adding Independent Uncertainties

1. Attach the vectors at the dot
2. σ_a is fixed in space
3. σ_b is free to rotate about the dot



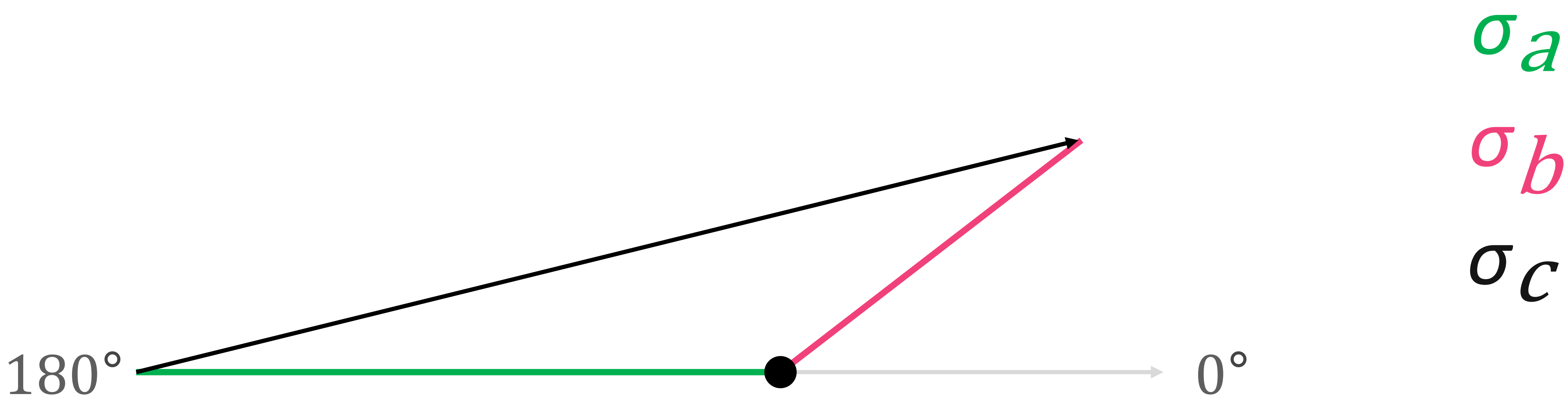
Adding Independent Uncertainties

1. Attach the vectors at the dot
2. σ_a is fixed in space
3. σ_b is free to rotate about the dot



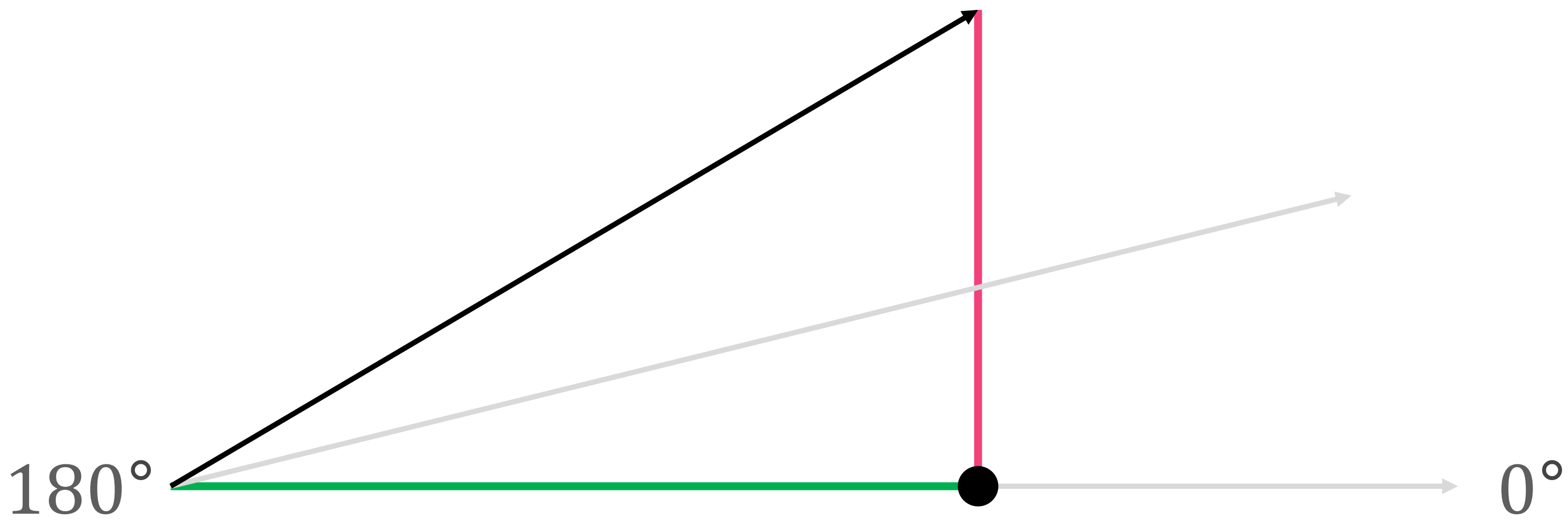
The rotation of σ_b in the plane traces out a circle.

Adding Independent Uncertainties



The length of the cumulative uncertainty, σ_c, is the shortest distance between the initial endpoint of σ_a and the final endpoint of σ_b.

Adding Independent Uncertainties



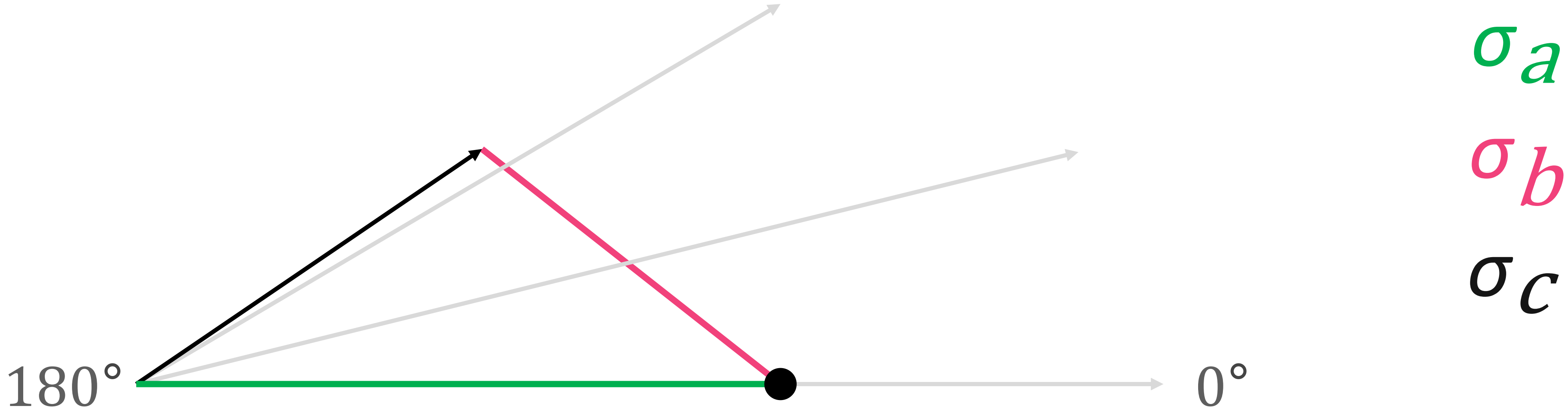
σ_a

σ_b

σ_c

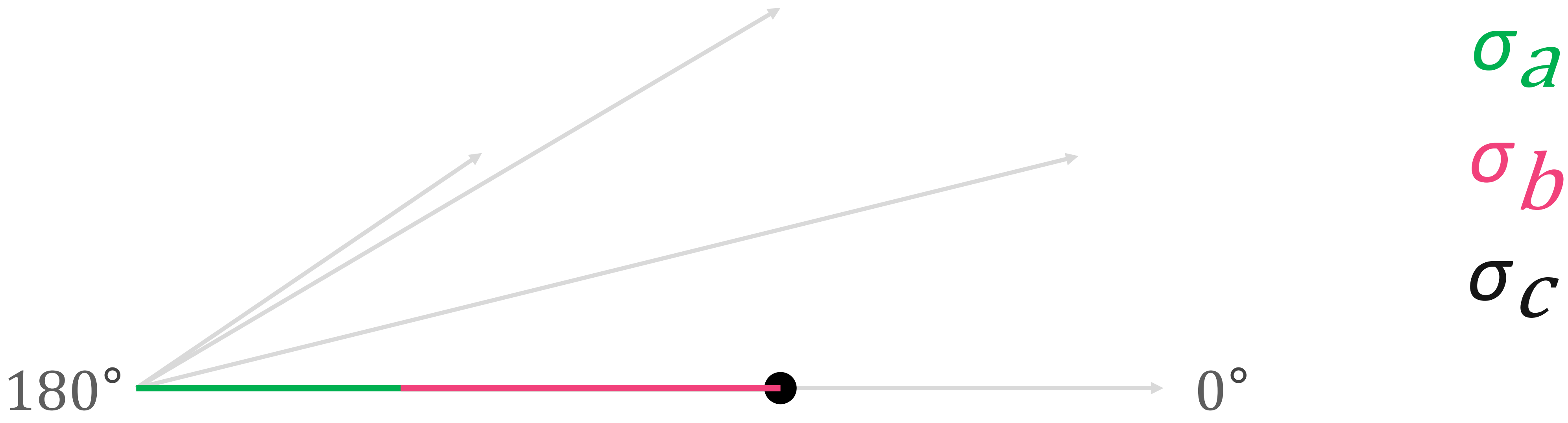
The length of the cumulative uncertainty, σ_c, is the shortest distance between the initial endpoint of σ_a and the final endpoint of σ_b.

Adding Independent Uncertainties



The length of the cumulative uncertainty, σ_c, is the shortest distance between the initial endpoint of σ_a and the final endpoint of σ_b.

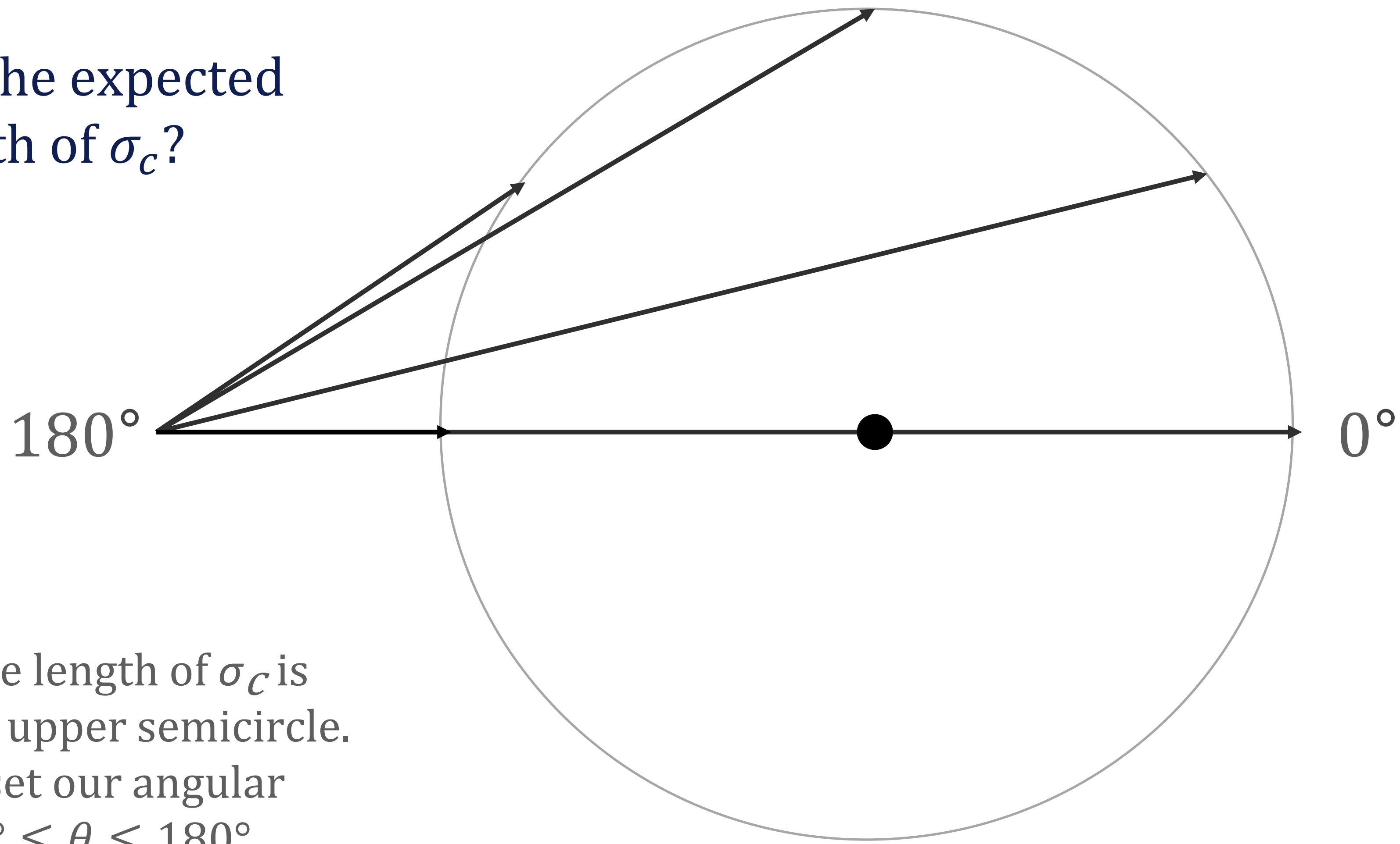
Adding Independent Uncertainties



The length of the cumulative uncertainty, σ_c, is the shortest distance between the initial endpoint of σ_a and the final endpoint of σ_b.

Adding Independent Uncertainties

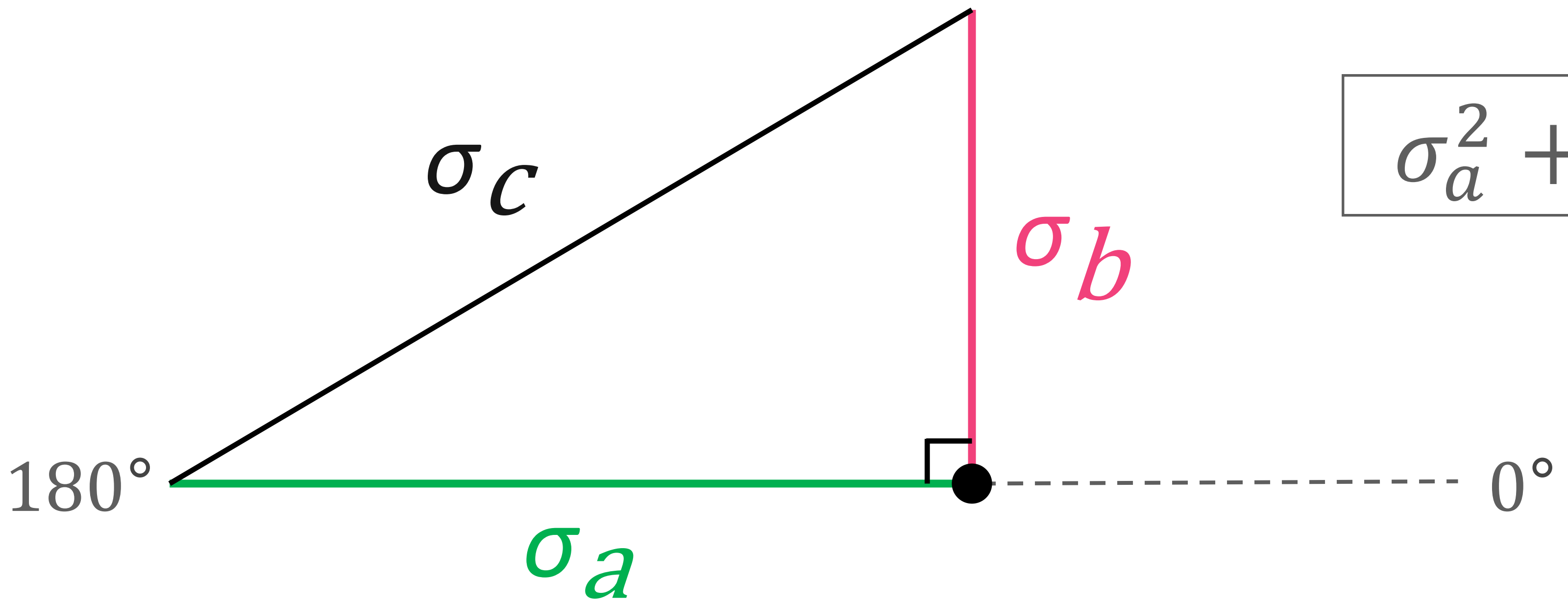
What is the expected length of σ_c?



σ_C

Every accessible length of σ_C is specified in the upper semicircle. Therefore, we set our angular bounds to be $0° \leq \theta \leq 180°$.

Adding Independent Uncertainties



$$\sigma_a^2 + \sigma_b^2 = \sigma_c^2$$

Since the uncertainties are independent of one another, σ_b is like a fair spinner, where **every landing position is equally probable**. Therefore, the expected landing position is 90°, the average of the upper and lower angular bounds.

For independent uncertainties, σ_c is the hypotenuse of a right triangle, which is solved by the Pythagorean Theorem.

Takeaways: Case of Independence

Value of θ	Description of child level uncertainties	Description of magnitude of cumulative uncertainty
0°	Aligned, superimposing	Upper bound
90°	Perpendicular	Expected value
180°	Anti-aligned, canceling	Lower bound

- Independent uncertainties are uncommon in cost estimating
- Simplified case sets up the correlated one

Part 4: Correlated Uncertainties and the Law of Cosines

Correlated Random Variables

$$Var(Cost) = \sum_{i=1}^{n} \sigma_i^2 + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \rho_{X_i,X_j} \sigma_i \sigma_j$$

Variance of sum of correlated variables

$$Var(Cost) = \sigma_1^2 + \sigma_2^2 + 2\rho_{X_1,X_2}\sigma_1\sigma_2$$

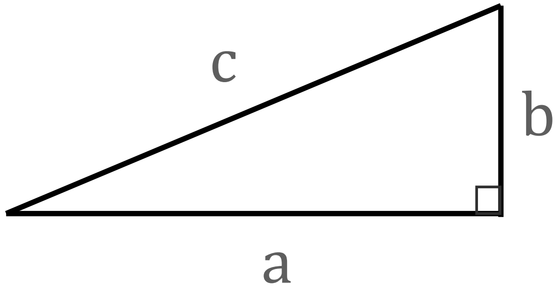
Reduce to 2D case

$$c^2 = a^2 + b^2 + 2ab\cos\theta$$

Law of Cosines

Geometry Refresher

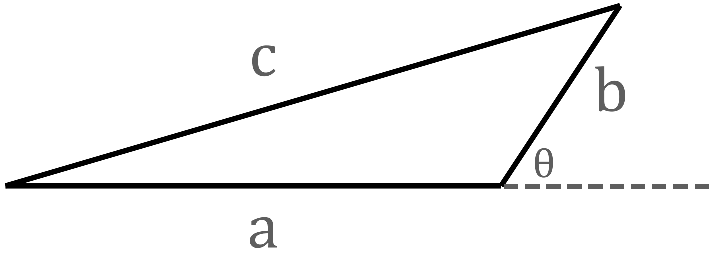
Pythagorean Theorem



$$a^2 + b^2 = c^2$$

Only works on right triangles

Law of Cosines



$$c^2 = a^2 + b^2 + 2ab\cos\theta$$

Generalizes the Pythagorean Theorem to arbitrary triangles

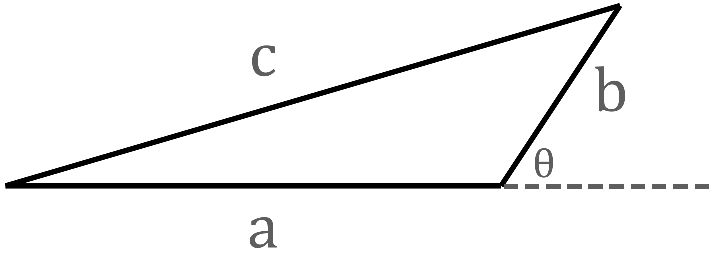
Geometry Refresher

Where does this term come from?

Why is θ outside of the triangle?

Vector Algebra

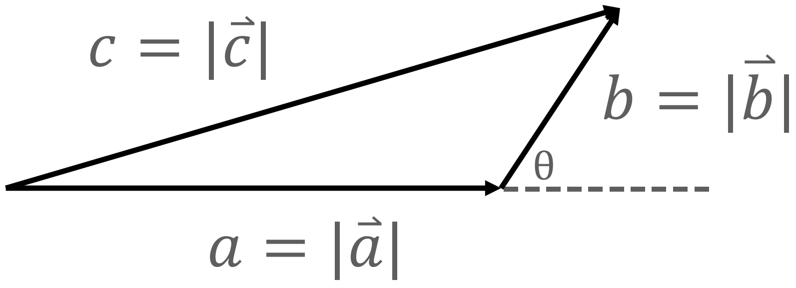
Law of Cosines



$$c^2 = a^2 + b^2 + 2ab\cos\theta$$

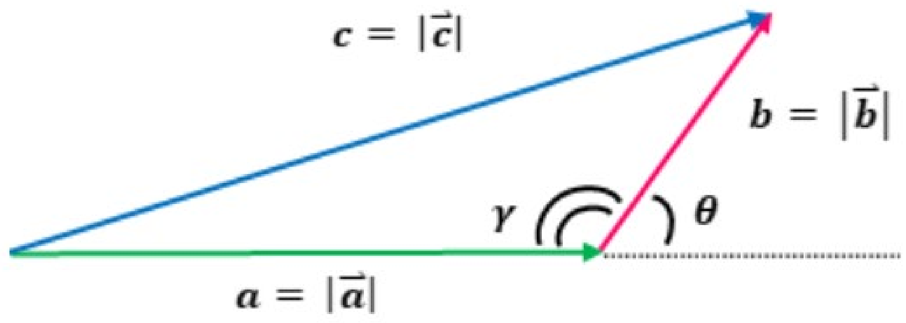
Generalizes the Pythagorean Theorem to arbitrary triangles

Deriving the Law of Cosines from Vector Addition



Vector addition	$\vec{c} = \vec{a} + \vec{b}$
Square both sides	$(\vec{c})^2 = (\vec{a} + \vec{b})^2$
Dot product	$\vec{c} \cdot \vec{c} = (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b})$
Distributive property	$\vec{c} \cdot \vec{c} = (\vec{a} \cdot \vec{a}) + (\vec{b} \cdot \vec{b}) + (\vec{a} \cdot \vec{b}) + (\vec{b} \cdot \vec{a})$
$\vec{u} \cdot \vec{v} = \lvert\vec{u}\rvert\lvert\vec{v}\rvert \cos\varphi$	$\lvert\vec{c}\rvert^2 \cos(0°) = \lvert\vec{a}\rvert^2 \cos(0°) + \lvert\vec{b}\rvert^2 \cos(0°) + 2\lvert\vec{a}\rvert\lvert\vec{b}\rvert \cos(\theta)$
Law of Cosines	$\lvert\vec{c}\rvert^2 = \lvert\vec{a}\rvert^2 + \lvert\vec{b}\rvert^2 + 2\lvert\vec{a}\rvert\lvert\vec{b}\rvert \cos(\theta)$

θ Describes the Alignment between Vectors



$$\gamma + \theta = 180°$$

$$\vec{a} + \vec{b} = \vec{c}$$

- θ signifies angular displacement from perfect alignment
- A small rotation in θ causes a small deviation in the direction of the vectors
- In statistics, we use another word to describe the “alignment” of two variables…

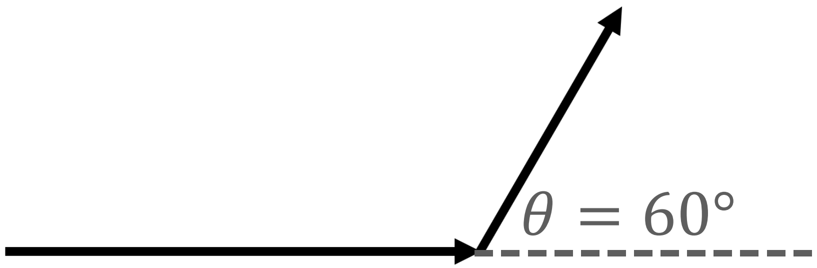
Correlation

- Correlation quantifies the strength of the association between two variables

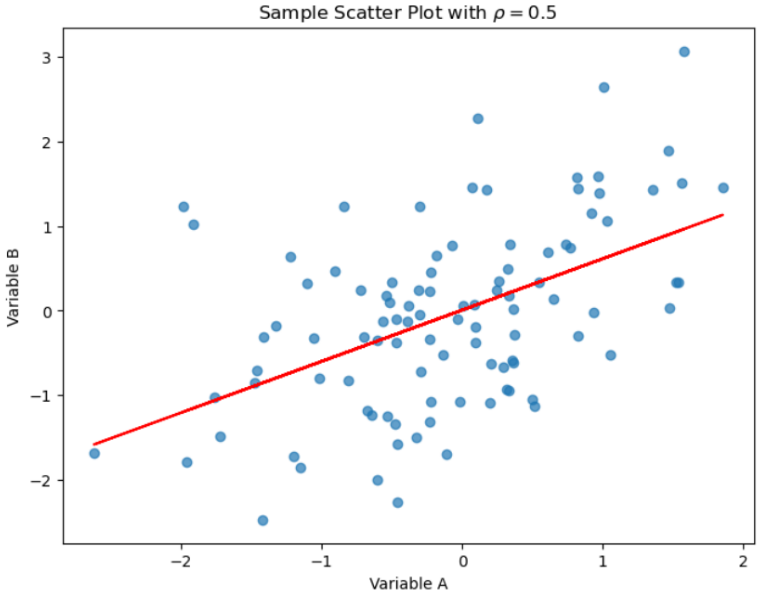
$$Var(A+B) = {\sigma_a}^2 + {\sigma_b}^2 + 2\sigma_a\sigma_b(\rho)$$

$$c^2 = a^2 + b^2 + 2ab(\cos\theta)$$

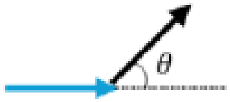
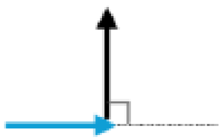
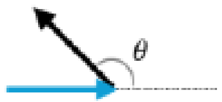
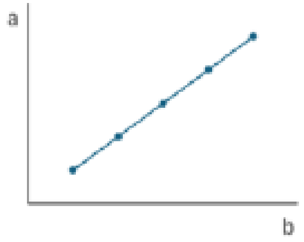
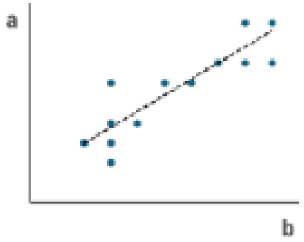
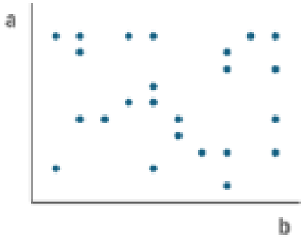
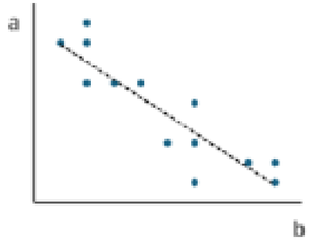
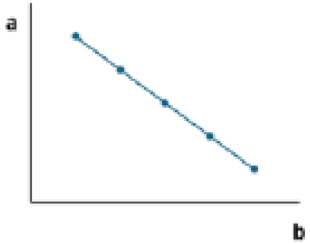
$$\rho = \cos\theta$$



$$\rho = \cos 60° = 0.5$$



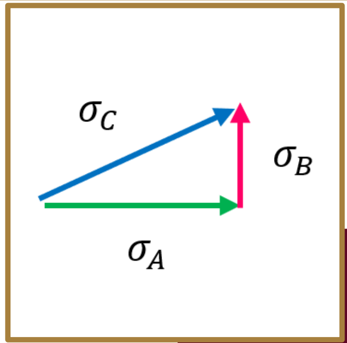
Connecting Uncertainty Vectors and Correlation

	Case A	Case B	Case C	Case D	Case E
Uncertainty Vectors & Angle of Rotation					
θ	$0°$	$0° < \theta < 90°$	$90°$	$90° < \theta < 180°$	$180°$
$\rho = \cos\theta$	1	$1 > \rho > 0$	0	$0 > \rho > -1$	-1
Relationship between Variables	Perfect Positive Correlation	Positive Correlation	Independent, No Correlation	Negative Correlation	Perfect Negative Correlation
Plot of Variables					

Applications in ACE

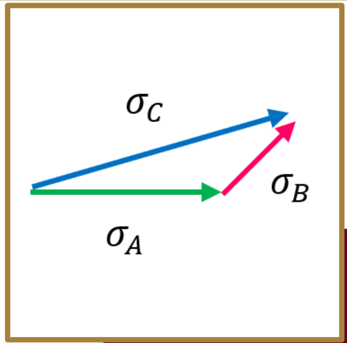
- ACE uses correlation in the case of dependence between two variables
  - **Functional Correlation:** Implicit correlation built in through function of the model
  - **Defined Correlation:** Applied correlation using the 'Group' and 'Group Strength' columns typically using the heuristic approach (subjective assessment)
- ACE and the USAF handbook provide correlation guidance
  - Able to verify subjective assessment and guidance through utilizing the law of cosines
- During the Monte Carlo simulation, the Group Strength Algorithm aims to:
  - Use the dominant row, series of Latin Hypercube draws, & positional ranking to maintain the original distribution of the task
  - Capture the correlation between the dependent and dominant tasks
  - Prevent unwanted correlations of dependent tasks within the same group

Summary



Independent Uncertainties

- Correlation coefficient = 0
- Angular separation = 90°
- Child-level uncertainties are added
- Pythagorean Theorem



Correlated Uncertainties

- Correlation coefficient ≠ 0
- 0° ≤ Angular separation < 90°

or

- 90° < Angular separation ≤ 180°
- Sum of correlated uncertainties
- Law of Cosines

Questions?

Backup

References

- [1] U.S. Government Accountability Office, GAO-20-195G Cost Estimating and Assessment Guide, Washington, DC: U.S. Government Accountability Office, p. 140, 2020.
- [2] S. Koellner, "Spread Too Thin: Managing Coefficient of Variation in Monte-Carlo Based Cost Models," ICEAA 2023 Professional Development & Training Workshop, p. 29, 2023.
- [3] G. Vernez, H. G. Massey, “The Acquisition Cost-Estimating Workforce,” RAND Corporation, p. 9, 2009.
- [4] P. R. Garvey, Probability Methods for Cost Uncertainty Analysis, Bedford, MA: Marcel Dekker, Inc., p. 182, 2000.
- [5] G. White, T. Sikorski, J. Landay, M. Ahmed, “Limiting Case Analysis in an Electricity and Magnetism Course,” American Physical Society, 2023.
- [6] Tecolote Research, Inc., U. S. Air Force Cost Risk and Uncertainty Analysis Handbook, p. 23, 2007.
- [7] Tecolote Research, Inc., ACE 8.2 Help Guide, 2024.

Deriving the Law of Cosines from Vector Subtraction

$$\vec{c} = \vec{a} - \vec{b}$$

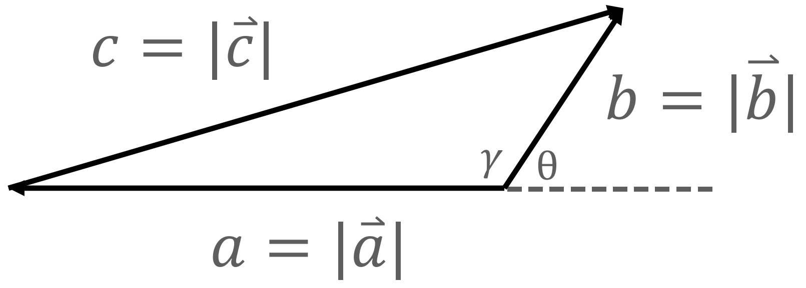
$$(\vec{c})^2 = (\vec{a} - \vec{b})^2$$

$$\vec{c} \cdot \vec{c} = (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b})$$

$$\vec{c} \cdot \vec{c} = (\vec{a} \cdot \vec{a}) + (\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{b}) - (\vec{b} \cdot \vec{a})$$

$$|\vec{c}|^2 \cos(0°) = |\vec{a}|^2 \cos(0°) + |\vec{b}|^2 \cos(0°) - 2|\vec{a}||\vec{b}| \cos(\gamma)$$

$$|\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}| \cos(\gamma)$$



$$\gamma + \theta = 180°$$
$$\cos\theta = -\cos\gamma$$

Application – Pythagorean Theorem

- Suppose the total cost of a system is given by Cost = X_1 + X_2 . Let X_1 denote the cost of the system's prime mission product. The distribution of X_1 is normal with a mean of 30 $M and a standard deviation of 12 $M. Let X_2 denote the cost of system testing. The distribution of X_2 is normal with a mean of 20 $M and a standard deviation of 5 $M. Suppose X_1 and X_2 are independent random variables. Determine the expected value of the total project cost and the total project standard deviation.

- Since the central tendencies are scalar quantities, the calculation of E(Cost) is straightforward:

$$\mathbf{E(Cost) = E(}\boldsymbol{X_1}\mathbf{) + E(}\boldsymbol{X_2}\mathbf{) = 30\ \$M + 20\ \$M = 50\ \$M}$$

- To sum together the standard deviations, we apply:

$$\boldsymbol{\sigma_{total} = \sqrt{\sum_{i=1}^{n} \sigma_i^2}}$$

- Where n is the number of child-level standard deviations. Here all the values in $\{\sigma_i\}$ are independent from one another. Setting n = 2 and plugging in the given quantities, we have:

$$\boldsymbol{\sigma_{total} = \sqrt{\sigma_1^2 + \sigma_2^2} = \sqrt{(12\ \$M)^2 + (5\ \$M)^2} = 13\ \$M}$$

- The expected value for the total program cost is 50 $M with a standard deviation of 13 $M

Application – Law of Cosines

- Suppose the total cost of a system is given by Cost = $X_1 + X_2$. Let X_1 denote the cost of the system's prime mission product. The distribution of X_1 is normal with a mean of 30 $M and a standard deviation of 12 $M. Let X_2 denote the cost of system testing. The distribution of X_2 is normal with a mean of 20 $M and a standard deviation of 5 $M. Suppose X_1 and X_2 are positively correlated with correlation coefficient $\rho = 0.5$. Determine the expected value of the total project cost and the total project standard deviation.

- Since the central tendencies are scalar quantities, the calculation of E(Cost) is straightforward:

$$\mathbf{E(Cost) = E(X_1) + E(X_2) = 30\ \$M + 20\ \$M = 50\ \$M}$$

- To sum together the standard deviations, we apply:

$$Var(Cost) = \sum_{i=1}^{n} \sigma_i^2 + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \rho_{X_i,X_j}\sigma_i\sigma_j$$

- The equation supports the summation of an arbitrary number, *n*, of correlated standard deviations. Setting *n* = 2 and plugging in the given quantities, we have:

$$Var(Cost) = \sigma_1^2 + \sigma_2^2 + 2\rho_{X_1,X_2}\sigma_1\sigma_2$$

$$Var(Cost) = (12\ \$M)^2 + (5\ \$M)^2 + 2(0.5)(12\ \$M)(5\ \$M) = 229\ \$M^2$$

$$\sigma_{total} = \sqrt{Var(Cost)} = \sqrt{229\ \$M^2} \approx 15.13\ \$M$$

- The expected value for the total program cost is 50 $M with a standard deviation of approximately 15.13 $M. Notably, the expected value for the mean is 50 $M regardless of whether the variables are independent or correlated; however, the total uncertainty increases from 13 $M in the independent case to ~15.13 $M in the positively correlated case.

Abstract

Cost estimators require a deep understanding of risk and uncertainty to produce reliable, quality cost models. Since analysts come from diverse academic fields, they may not possess the intersectional knowledge of statistics and linear algebra that manifests in uncertainty analysis. The objective of this paper is to demonstrate this intersection using the intuitive language of geometry: we reduce the abstract problem of determining cumulative uncertainty to the more tangible problem of solving a triangle. The authors leverage the Pythagorean theorem and the Law of Cosines to geometrically explain the summations of independent and linearly correlated standard deviations, respectively. By adopting this theoretical framework for uncertainty analysis, estimators can establish a foundation from which to develop an understanding of other advanced analytical methods at the frontier of cost estimation.