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Here, There Be Dragons: Considering the Right Tail in Risk Management

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The "portfolio effect" is a common designation of a supposed reduction of cost risk achieved by funding multiple projects (the "portfolio") that are not perfectly correlated with one another. It is often relied upon in setting confidence-level policy for program or organization budgets that are intended to fund multiple projects. The idea of a portfolio effect has its roots in modern finance, as pioneered by 1990 Nobel Memorial Prize in Economic Sciences recipient Harry Markowitz (1959). On the other hand, in presentations to four recent ISPA-SCEA conferences, 2007–2010, the present author argued that, when applied to Government budgeting, the portfolio effect is more myth than fact. However, current National Aeronautics and Space Administration and Department of Defense policy guidance relies heavily upon this apparently chimerical effect. The objective of the present article is to propose a superior alternative budgeting decision process based on a concept called "conditional tail expectation" that better measures project risk exposure in terms of the project's expected shortfall in funding. Also called "tail value at risk," use of this risk-assessment technique is growing in popularity in a variety of financial contexts, including insurance.

Introduction

Jorion (2007) wrote that "Western Europe conquered the world because of a technological revolution that started because of attempts to measure the world." In the same way, attempts to measure risk more definitively, realistically, and accurately will surely lead to better project management.

Current government policy guidance, particularly that offered by National Aeronautics and Space Administration (NASA) and Department of Defense (DoD), calls for setting funding at a specific percentile, typically the 70th or 80th. The optimistic belief underlying this guidance is that the portfolio effect will allow total organization-wide funding to support a much higher degree of confidence, perhaps even one above the 90th percentile. However, if it turns out that a portfolio effect of the kind envisioned does not really occur, policy guidance that recommends funding at a relatively low percentile, like the 70th or 80th, will result in numerous overruns, insufficient reserves, and other financial difficulties at the organizational level. Funding at such "low" levels might very well result in overruns for 20–30% of projects, so cost growth will continue to be a predictable occurrence.

Furthermore, funding at a specific percentile provides no insight into how much extra funding may be needed in reserves. Depending upon the variance of the cost risk distribution and its other statistical characteristics, such as skewness and kurtosis, the needed amount of extra funding can vary significantly. Thus the right tail (the portion of a probability distribution in the region of the higher percentiles) must be taken into

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consideration when establishing reserves. A more informative metric, called "conditional tail expectation" (CTE), has been proposed by analysts working in finance and insurance to measure this expected shortfall. Another name for that metric is "tail value at risk."

Cost-related approaches to risk management for NASA and other government projects seem to focus solely on finding a single percentile to budget against. NASA policy in particular explicitly mentions the 70 and 50% levels, for example, and some government organizations call for budgeting to the 80th percentile. Furthermore, other U.S. Government organizations tend to follow the lead of NASA and other organizations on this issue. In the financial industry, a percentile level is referred to as the "Value at Risk" (VaR) and defined as the maximum possible loss at a given probability level (Embrechts et al., 2005). In this article, we will consider the terms percentile level and VaR to be interchangeable.

In mathematical terms, suppose that $\boldsymbol{C}$ is a random variable representing project cost, and $\boldsymbol{F_C(x) = P(C \le x)}$ is its probability distribution function. Then the $\boldsymbol{VaR}$ of $\boldsymbol{C}$ at probability level $\boldsymbol{\alpha}$ is technically defined as

$$\begin{aligned} \boldsymbol{VaR_\alpha(C) = Q_\alpha(C) = F_c^{-1}(\alpha) = inf\{x{:}F_c(x) \ge \alpha\}} \\ \boldsymbol{= inf\{x{:}\ 1 - F_c(x) \le 1 - \alpha\}}, \end{aligned}$$

where $\boldsymbol{Q_\alpha(C)}$ is the $\boldsymbol{100\alpha}$th percentile of $\boldsymbol{C}$, $\boldsymbol{F^{-1}}$ denotes the inverse function of $\boldsymbol{F}$, and $\boldsymbol{inf\{x{:}statement\}}$ signifies the smallest value of $\boldsymbol{x}$ for which the statement following the colon is true. That is, VaR is a percentile of the cost risk distribution. For example, for $\boldsymbol{\alpha = 0.95}$, the 95th percentile of a normal distribution with mean equal to \$600 and standard deviation equal to \$200 is approximately \$929, so $\boldsymbol{VaR_{0.95} = \$929}$. In this article, the terms $\boldsymbol{VaR_\alpha}$, $\boldsymbol{Q_\alpha}$, and $\boldsymbol{F^{-1}(\alpha)}$, for a specified value of $\boldsymbol{\alpha}$, are used interchangeably.

Percentile (or VaR) funding has some merits. It can be used to compare the funding requirements of different projects. It can be easily understood by decision makers who may not be fluent in the details of probability and statistics. Risk reserves can be defined in terms of it. It is currently part of NASA and DoD policies and is commonly used even in the banking industry. However risk management shouldn't stop at that point because, for example, even funding at the 70th percentile means that there is almost a one-in-three chance of experiencing an overrun. And not only that, but funding at the 70th percentile provides no information about what may happen above that level. If the 70th percentile is exceeded, how much additional funding is expected to be required?

Funding at any set percentile is budgeting in the dark, ignoring the truly risky, bad-day events that projects, or at least portfolios, should have funds available to pay for when they occur. Funding against a low percentile is like whistling in the dark, hoping that right-tail events do not occur. Funding against a high percentile is unfortunately not that much different, even though we think we have better prepared for right-tail events. And not all right tails are created equal. Some distributions, such as the triangle, have no tails (a subcategory of "thin"), the normal and lognormal have thin tails, but others, typically those which are more realistic for projects, have fat tails. As on old maps that depicted dragons and other monsters in uncharted territory (see Figure 1), who knows what risks lurk beyond the 70th percentile?

Consider four different distribution types, each of which has the same 70th percentile, but different characteristics, as displayed in Figure 2. In Figure 2, four distributions, a triangular, normal, lognormal, and Pareto, are all displayed. Each has different characteristics, and their defining parameters are listed in Table 1.

FIGURE 1 A section of the Carta Marina by Olaus Magnus, 16th Century (color figure available online).

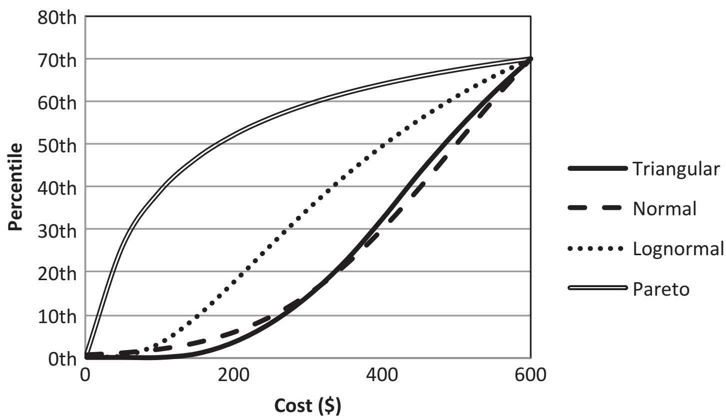

FIGURE 2 Four different distributions with the same 70th percentile.

TABLE 1 Defining parameters of the four distributions

Distribution type	Defining parameters		
Triangular	$L = 100.00$	$M = 407.41$	$H = 1{,}000.00$
Normal	$\mu = 500.00$	$\sigma = 190.69$	
Lognormal	$\mu = 6.0000$	$\sigma = 0.7569$	
Pareto	$\alpha = 0.50$	$\theta = 59.34$	

While the triangular, normal, and lognormal distributions are usually well understood by cost analysts, the two-parameter Pareto distribution may be unfamiliar to many. Its cumulative distribution function is given by the following expression:

$$F_{Par}(x) = Pr\{Par \le x\} = 1 - \left(\frac{\theta}{x + \theta}\right)^{\tau} \text{ for } x \ge 0$$

$$= 0 \text{ for } x < 0.$$

The Pareto distribution, not commonly used in cost analysis, is an example of a fat-tailed distribution, which has received a great deal of attention in the financial press in recent years (see Taleb, 2007, for an example). So it is of interest to the cost community to compare what a fat-tailed distribution may mean in practice for setting risk reserves, compared with the consequences of modeling on the basis of the thin-tailed triangular, normal, and lognormal distributions that are often seen in practice in the majority of cost estimates.

Although each distribution in Table 1 happens to have the same 70th percentile, as shown in Figure 2, should we expect them to have similar tail behavior? The obvious answer is no, and a cursory glimpse at the S-curves' right-tail characteristics, as displayed in Figure 3, make this plain.

The triangular and normal distributions both have thin tails. There is relatively little risk beyond the 70th percentile, and while the lognormal has a slightly fatter tail, the Pareto has an extremely fat tail. The 70th percentile for all four distributions is \$600. But to get to the 80th percentile, an additional \$60 is needed for the normal, \$80 for the triangular, but \$170 is needed for the lognormal, and a whopping \$830 more is needed for the Pareto. A comparison of the tail behavior of the four distributions is shown in Table 2.

Table 2 shows there is significant tail risk when the lognormal or the Pareto is an appropriate model for the cost distribution. Indeed the 99th percentile for the Pareto is 1,000 times greater than its 70th percentile, indicating enormous tail risk. Should four different projects whose cost distributions follow the four different risk profiles seen in this example, be funded at the same level? That is what would happen in the case of 70th percentile confidence funding, but funding each of two projects at \$600, one of which

TABLE 2 Comparison of tail percentiles for the four distributions

Percentile	Triangular	Normal	Lognormal	Pareto
70th	\$600	\$600	\$600	\$600
80th	\$680	\$660	\$763	\$1,424
90th	\$770	\$744	\$1,064	\$5,875
95th	\$840	\$814	\$1,401	\$23,677
99th	\$930	\$944	\$2,347	\$593,341

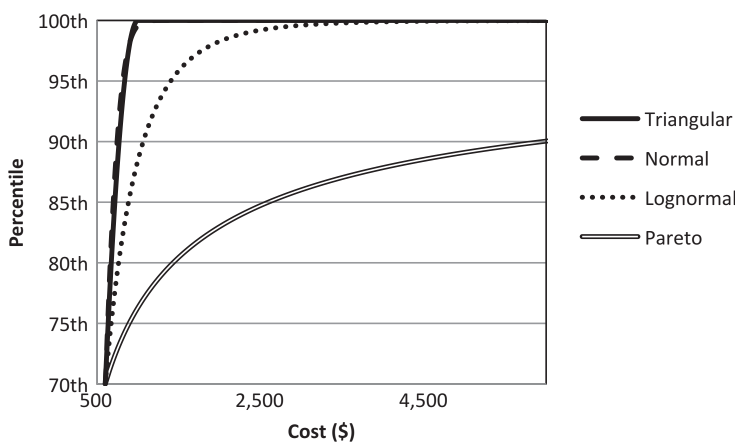

FIGURE 3 Tail behavior of the four distributions.

follows the triangular distribution and the other the Pareto, will have significantly different consequences in the 30% of those cases in which the projects' 70th percentiles are exceeded.

The Pareto distribution in this example may not model the cost-risk behavior of an actual Government project, because almost any such project would surely be cancelled long before experiencing cost growth in the range of a several-hundred-fold increase. Just imagine the prospect of a billion-dollar project growing to a trillion-dollar project! However, the Pareto distribution is illustrative of the real magnitude of risk seen in financial markets and in catastrophic risks for natural disasters such as the massive oil spill in the Gulf of Mexico in 2010. These kinds of risks are what financial writers like Taleb (2007) call "black swan" events. Risk magnitudes that arise from such distributions, namely the kind typically referred to as fat-tailed in the field of finance, make costs overruns of aerospace and defense projects seem puny by comparison.

And does percentile budgeting even truly amount to effective risk management policy? Suppose you are shopping for a new car. You mention that safety is your top concern. The salesman says he has a great, safe car available. You ask about the air bags. The salesman answers, "Of course the air bags work! Seventy percent of the time they work fine. Only 30% of the time, the air bags will fail to deploy." Would you buy such a car? According to hedge fund manager David Einhorn, "Risk management is the air bag that must always work, but only in the multi-sigma event where you have an accident" (Brown and Einhorn, 2008). This is the complete opposite of percentile budgeting to manage risk. Merrill Lynch, in its September 28, 2002 10-Q filing, stated "VaR [*aka* percentile] measures do not convey the magnitude of extreme events." (Triana, 2009). Thus percentile budgeting is risk management based on misleading information. As Taleb stated in his book, *The Black Swan*, "You're worse off relying on misleading information than on not having any information at all. If you give a pilot an altimeter that is sometimes defective he will crash the plane. Give him nothing at all and he will look out the window." The point is that average boring laid-back events should not be the program manager's only focus, but that's all 70th percentile budgeting protects against.

Coherent Risk Measures

A risk measure quantifies exposure to risk. When applied to cost risk, it is a single number that is used to represent the cost risk of a project or program. In this article, risk measures will be denoted by the function $\boldsymbol{\rho}$. A risk measure for a project's cost $\boldsymbol{X}$, which is denoted $\boldsymbol{\rho}(\boldsymbol{X})$, has the same units as those used to measure cost. That is, if cost is measured in dollars then $\boldsymbol{\rho}(\boldsymbol{X})$ is also measured in dollars.

The variance of a distribution is one example of a risk measure, since it quantifies the spread in the cost distribution. VaR, namely a percentile, is a more informative risk measure, and there are many others that have been and are being used. How do we know which risk measures are "good" and which are not? In other words, what properties should a risk measure have? This issue has been studied for the insurance industry specifically and for risk measurement in general. In a groundbreaking paper, Artzner et al. (1999) introduced the notion of "coherence" of risk measures.

One property important for a risk measure is that when two random variables are combined, the portfolio of the two corresponding projects should be no riskier than the sum of the individual projects' risk measures. This property is expressed algebraically by stating that a risk measure $\boldsymbol{\rho}$ should have the *subadditivity* property, namely, that

$$\boldsymbol{\rho}(\boldsymbol{X}+\boldsymbol{Y}) \leq \boldsymbol{\rho}(\boldsymbol{X})+\boldsymbol{\rho}(\boldsymbol{Y}).$$

This property ensures that there should be some diversification benefit from combining risks from separate projects into a portfolio. A better-known term for subadditivity is the "portfolio effect" (Anderson, 2004), which is being relied upon by policymakers in setting funding levels of individual projects to relatively low levels, such as the 70th percentile and below, and expecting the portfolio to automatically benefit from a higher level of confidence.

A second desirable property of risk measures is that, if one cost ($\boldsymbol{X}$) is always higher than a second cost ($\boldsymbol{Y}$), then the risk measure of $\boldsymbol{X}$ should be higher than the risk measure for $\boldsymbol{Y}$. For example, if the cost of structures hardware is higher in every circumstance than the cost of thermal control hardware, then the 70th percentile of the cost risk distribution should be higher for the structures subsystem than for the thermal control subsystem. This is the property of *monotonicity*, and can be stated in equation form as

$$\boldsymbol{X \leq Y \textbf{ for all possible outcomes } \Rightarrow \rho(X) \leq \rho(Y)}.$$

The symbol "⇒" means "implies."

A third desirable property is that the risk measure should be invariant in respect to the currency in which the risk is measured or whether cost is accounted for in thousands or millions of dollars. It also includes the characteristic that an equivalent increase or decrease in exposure to the risk requires an equivalent change in the amount of capital needed to guard against this risk. This is the property of *positive homogeneity*, which can be expressed as $\boldsymbol{\rho(cX) = c\rho(X)}$ for any constant real number $\boldsymbol{c}$. This property makes sense because risk characteristics should not change based on the unit in which the risk is measured. For example, the only difference between risk reserves being measured in thousands vice millions of dollars should be that when measured in thousands of dollars, the risk reserves should be multiplied by 1,000, that is, the risk is \$5 million, or 5,000 thousands of dollars. Another example would be the difference between risks expressed in dollars versus euros should be only the currency conversion (ignoring exchange rate risk over time).

Also important in measuring risk is the realization that a project may consist of some aspects that are risky and some that are not. If a project consists of a portion that is subject to risk and a portion that is not subject to risk, then the cost of the portion subject to risk is a random variable but the cost of the portion not subject to risk is a fixed, constant amount. That is, if there is a stochastic component and a deterministic component, the total risk reserve should be the sum of an amount determined by the risk of the non-deterministic component and the fixed, deterministic amount. This property is called *translation invariance* and can be expressed as $\boldsymbol{\rho(X+c) = \rho(X) + c}$. An example of this would be a project that includes a firm-fixed price (FFP) contract for a single avionics component. The rest of the system will have risk, but from the Government's perspective, the FFP contract bears no risk. So if the 80th percentile of the risk distribution for the rest of the system is \$70 million, and the FFP contract for the avionics component is \$10 million, then the total 80th percentile is $\$70 + \$10 = \$80$ million.

A coherent risk measure is defined as a risk measure $\boldsymbol{\rho(X)}$ that has the four properties of *subadditivity*, *monotonicity*, *positive homogeneity*, and *translation invariance*.

A simple and popular risk measure is defined as the mean plus a fixed number of standard deviations, i.e., $\boldsymbol{\mu + k\sigma}$, in particular $\boldsymbol{\rho(X) = \mu(X) + k\sigma(X)}$, which is called the *standard deviation principle*.

The standard deviation of a sum of random variables $\boldsymbol{X}$ and $\boldsymbol{Y}$ is

$$\boldsymbol{\begin{aligned}\sigma^2(X+Y) &= \sigma^2(X) + \sigma^2(Y) + 2\lambda\sigma(X)\sigma(Y) \\ &\leq \sigma^2(X) + \sigma^2(Y) + 2\sigma(X)\sigma(Y) = (\sigma(X) + \sigma(Y))^2.\end{aligned}}$$

Because the correlation λ between X and Y is such that $-1 \leq \lambda \leq 1$. This implies that $\sigma(X+Y) \leq \sigma(X)+\sigma(Y)$, and so that

$$\begin{aligned}\rho(X+Y) &= \mu(X+Y)+k\sigma(X+Y)\\ &= \mu(X)+\mu(Y)+k\sigma(X+Y)\\ &\leq \mu(X)+\mu(Y)+k(\sigma(X)+k\sigma(Y))\\ &= \mu(X)+k\sigma(X)+\mu(Y)+k\sigma(Y)\\ &= \rho(X)+\rho(Y).\end{aligned}$$

This discussion proves that the standard deviation principle is *subadditive*.

Also, the standard deviation principle is *positive homogeneous*, since

$$\begin{aligned}\mu(c(X+Y))+k\sigma(c(X+Y)) &= c\mu(X+Y)+ck\mu(X+Y)\\ &= c(\mu(X+Y))+(k(\sigma(X+Y))).\end{aligned}$$

And since standard deviation is not affected by a translation of the random variable, although the mean is shifted by exactly the translation, the standard deviation principle is *translation invariant*.

However, the standard deviation principle is *not monotonic*. To see this, consider a bivariate random variable defined as

$$p(X,Y) = \begin{cases} \mathbf{0.25\,for\,X = 0,\ Y = 4} \\ \mathbf{0.75\,for\,X = 4,\ Y = 4} \end{cases}.$$

In this case, $\mu_X = 3$, $\mu_Y = 4$, $\sigma_X = \sqrt{3}$, $\sigma_Y = 0$. Note that even though $X \leq Y$ we have that $\mu_X+\sigma_X = 3+\sqrt{3} > 4 = 4+0 = \mu_Y+\sigma_Y$. It follows that the standard deviation principle is not monotonic when $k = 1$, and this one counterexample (of course, there are many more) serves to prove its non-monotonicity.

Risk as measured according to the standard deviation principle is not equivalent to risk as measured according to the *VaR* metric, unless we restrict our attention to normally distributed random variables. In that case the *VaR* metric is a special case of the standard deviation principle with k set to satisfy whichever percentile is selected. For the 70th percentile, $k \approx 0.5244$. Therefore, for normally distributed variables, *VaR* satisfies the conditions of translation invariance, positive homogeneity, and subadditivity by the same rationale as for the standard deviation principle. But also a percentile measure is *monotonic* regardless of the distribution involved since, if $Pr(Y \leq x) = \beta$ and $Pr(X \leq Y) = 1$, then $Pr(X \leq x) \geq \beta$. Thus, in the special case of normally distributed random variables, both *VaR* and the standard deviation principle are coherent risk measures.

In general, however, *VaR* considered as a percentile of a cost distribution is translation invariant, monotonic, and positive homogeneous. But it is not guaranteed to be subadditive if non-normal random variables are involved. Indeed, there are cases in which percentile funding may be superadditive, i.e., cases for which $VaR(X+Y) \geq VaR(X) + VaR(Y)$. A mathematically intense example is described in Appendix A.

For a simple concrete example of superadditivity, suppose there are two projects in a portfolio. Each project has a budget equal to \$100, and there is a 25% probability of a

\$20 overrun. Each of the two projects is assumed to be independent. See Table 3 for the discrete probabilities.

Then the 75th percentile value for each project is \$120, while the 70th percentile is \$100, since $\boldsymbol{F(100) = 0.75}$ and \$100 is the smallest value in the set for which $\boldsymbol{F(100) > 0.70}$.

Since the two projects are independent, there is a $\mathbf{0.75 * 0.75 = 0.5625}$ probability that the total cost of the portfolio is equal to \$200. Also, there is a $\mathbf{2 * 0.75 * 0.25 = 0.375}$ probability that one project will have an overrun while the other will not, for a total portfolio cost equal to \$220. And there is a $\mathbf{0.25 * 0.25 = .0625}$ probability that both projects will experience a \$20 overrun, for a total portfolio value equal to \$240. The probabilities, along with the cumulative values, are displayed in Table 4.

The overall 70th percentile for the portfolio is \$220, since $\mathbf{F(220) > 0.70}$ and it is the smallest value in the set of outcomes with this property.

Thus the overall portfolio is riskier than the sum of the individual projects because the 70th percentile for the complete program exceeds the sum of the 70th percentiles of the individual projects. Even though common sense would tell you that managing two projects as a portfolio should be no riskier than managing them separately, and in fact should be less risky due to diversification benefits, measuring risks with percentiles can lead to the opposite conclusion. This provides two distinct examples which shows that not only does the portfolio effect not exist (Smart, 2008, 2009), for percentile funding, it is possible to have a reverse portfolio effect!

The previous example occurs frequently when the funding level is below the mean for individual projects. In the example above, funding was at \$100, but the mean outcome is equal to $\mathbf{\$100 * 0.75 + \$120 * 0.25 = \$105}$. For example, consider two independent normal distributions, each with mean $\boldsymbol{\mu}$ and standard deviation $\boldsymbol{\sigma}$. Suppose that funding for each project is set below the mean, at the $\boldsymbol{q}$th percentile. Since the normal distribution is symmetric, the 50th percentile is equal to the mean, so in this case, funding is also set below the 50th percentile. For the normal distribution, the percentile funding can be represented as $\boldsymbol{\mu + \phi^{-1}(q)\sigma}$, where $\boldsymbol{\phi^{-1}}$ is the inverse of the standard normal distribution. The sum of the percentile funding for each project is $\boldsymbol{2(\mu + \phi^{-1}(q)\sigma)}$.

Since the two normal distributions are independent (by assumption), the sum of the distributions is also normal with mean $\boldsymbol{2\mu}$ and standard deviation $\boldsymbol{\sqrt{\sigma^2 + \sigma^2} = \sqrt{2}\sigma}$.

TABLE 3 Potential costs and associated probabilities for the example

\$	Probability of Occurrence
100	75%
120	25%

TABLE 4 Portfolio probabilities and associated costs for the example

\$	Probability of Occurrence	Cumulative Probability
200	56.25%	56.25%
220	37.50%	93.75%
240	6.25%	100.00%

Percentile funding at the qth percentile for the portfolio is thus $2\mu + \sqrt{2}\phi^{-1}(q)\sigma$. Note that since the qth percentile is less than the 50th percentile that $\phi^{-1}(q)$ is less than zero since the mean of the standard normal distribution is equal to zero. Thus, it follows that

$$2(\mu + \phi^{-1}(q)\sigma) = 2\mu + 2\phi^{-1}(q)\sigma < 2\mu + \sqrt{2}\phi^{-1}(q)\sigma,$$

which means that the sum of the percentiles is less than the percentile of the sums, which demonstrates that there is a negative portfolio effect when normal distributions are funded below the mean (and also the 50th percentile).

As another concrete example, consider two independent lognormal distributions, one with mean 10, and standard deviation 2, and the other with mean 50 and standard deviation 20. The 50th percentiles for the two distributions are approximately 9.81 and 46.42, respectively. In this case, using Monte Carlo simulation to aggregate the two distributions, the 50th percentile of the sum of the distributions is found to be 56.46. This is slightly greater than the sum of the 50th percentiles of each lognormal distribution, which is approximately 56.23. Again, when funding below the mean, a negative portfolio effect is found to occur. Note that the example in Appendix A does not depend on setting the percentile level below the mean. Indeed the example in Appendix A shows a negative portfolio effect for any tail value.

Conditional Tail Expectation

Government agencies need to set policy on a course of action once the 70% mark is exceeded. Percentile funding is not a risk management policy. As we have discussed, it simply defines when a bad situation has occurred, but says nothing about what course of action to take once a bad event (cost has exceeded a specified percentile) has occurred. A better policy is one that would have specific funds set aside in the case of a specific bad event. One useful risk measure that comes in handy in such cases is the CTE. This is defined as the amount of cost growth to expect given that cost has exceeded a specified amount, that is

$$E\left[X | X > Q_\alpha\right],$$

where Q_α is a specified percentile. This risk measure is referred to the CTE. For example, $Q_{0.95}$ is by definition the 95th percentile. This risk measure is also called the "Tail Value at Risk" (TVaR) and "expected shortfall" (Embrechts et al., 2005). Using the term TVaR makes sense since, in the case of continuous cost distributions, it may be viewed as an integral of the right tail of the distribution:

$$CTE_\alpha = E\left[X | X > Q_\alpha\right] = \frac{1}{1 - F(Q_\alpha)} \int_{Q_\alpha}^{\infty} xf(x)dx = \frac{1}{1-\alpha} \int_{\alpha}^{1} VaR_u(X)du.$$

The latter equality can be derived by making the substitution $u = F(x)$ Note also that CTE_α can be written as

$$CTE_\alpha = E\left[X | X > Q_\alpha\right] = \frac{\int_{VaR_\alpha}^{\infty} xf(x)dx}{1-\alpha}$$

$$= VaR_\alpha - VaR_\alpha + \frac{\int_{VaR_\alpha}^{\infty} xf(x)dx}{1-\alpha}$$

$$= VaR_\alpha - VaR_\alpha \frac{\int_{VaR_\alpha}^{\infty} f(x)dx}{1-\alpha} + \frac{\int_{VaR_\alpha}^{\infty} xf(x)dx}{1-\alpha}.$$

Thus,

$$CTE_\alpha = VaR_\alpha + \frac{\int_{VaR_\alpha}^{\infty} (x - VaR_\alpha)f(x)dx}{1-\alpha}$$

$$= VaR_\alpha + \frac{\int_{VaR_\alpha}^{\infty} xf(x)dx - VaR_\alpha \left(\int_{VaR_\alpha}^{\infty} f(x)dx\right)}{1-\alpha}$$

$$= VaR_\alpha + \frac{E(X) - \int_0^{VaR_\alpha} xf(x)dx - VaR_\alpha \left(1 - \int_0^{VaR_\alpha} f(x)dx\right)}{1-\alpha}$$

$$= VaR_\alpha + \frac{E(X) - \int_0^{VaR_\alpha} xf(x)dx - VaR_\alpha(1 - F(VaR_\alpha))}{1-\alpha}$$

$$= VaR_\alpha + \frac{E(X) - (\int_0^{VaR_\alpha} xf(x)dx + VaR_\alpha(1 - F(VaR_\alpha)))}{1-\alpha}$$

$$= VaR_\alpha + \frac{E[X] - E[X_\alpha^{VaR}]}{1-\alpha}.$$

It is shown in Appendix B that, for a normal distribution,

$$CTE_\alpha(X) = \mu + \sigma \frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}$$

and, for a lognormal distribution, that

$$CTE_\alpha(X) = \frac{\mathbf{E[X]}\left[1 - \Phi\left(\frac{\ln \mathrm{VaR}_\alpha - \mu - \sigma^2}{\sigma}\right)\right]}{1-\alpha}.$$

For example, consider a single project whose cost risk has been modeled as a lognormal distribution with mean equal to \$100 million and standard deviation equal to \$50 million, $\mu = 4.49$, $\sigma = 0.72$, and the 70th percentile is equal to

$$e^{4.49 + z_{0.70} 0.72} \approx \$114.6\,\textit{million}.$$

Thus, in this instance,

$$CTE_{0.70} = 100 \cdot \frac{1 - \Phi\left(\frac{\ln 114.6 - 4.49 - 0.47^2}{0.47}\right)}{1 - 0.7} \approx \$\mathbf{159.7\ million.}$$

Therefore, given that the 70th percentile has been reached, the expected amount needed to complete the project will be \$160 million, roughly \$46 million above the 70th percentile budget. This is 40% more than that budgeted value.

In addition to providing a true risk management approach, in that the CTE establishes not only a trigger (the VaR event that occurs when cost grows to the percentile at which project funding has been set), but also an additional amount of reserves set aside in case bad times occur. CTE provides additional detail about the right tail relevant to a sensible risk management policy. And CTE is a coherent risk measure (Acerbi and Tasche, 2002). Most of the coherence properties follow naturally from properties of percentiles, in particular positive homogeneity and translation invariance. Monotonicity naturally follows, since if $\boldsymbol{X}$ is always less than or equal to $\boldsymbol{Y}$, the conditional expected value of $\boldsymbol{X}$ greater than some fixed value will always be less than the conditional expected value of $\boldsymbol{Y}$ for that same fixed value. To see that subadditivity holds, note that

$$\begin{aligned}
&\boldsymbol{CTE}_{\alpha}(\boldsymbol{X}) + \boldsymbol{CTE}_{\alpha}(\boldsymbol{Y}) - \boldsymbol{CTE}_{\alpha}(\boldsymbol{X}+\boldsymbol{Y}) \\
&\quad = \boldsymbol{E}[\boldsymbol{X}|\boldsymbol{X} > \boldsymbol{Q}_{\alpha}(\boldsymbol{X})] + \boldsymbol{E}[\boldsymbol{Y}|\boldsymbol{Y} > \boldsymbol{Q}_{\alpha}(\boldsymbol{Y})] - \boldsymbol{E}[\boldsymbol{X}+\boldsymbol{Y}|\boldsymbol{X}+\boldsymbol{Y} > \boldsymbol{Q}_{\alpha}(\boldsymbol{X}+\boldsymbol{Y})] \\
&\quad = \boldsymbol{E}[\boldsymbol{X}|\boldsymbol{X} > \boldsymbol{Q}_{\alpha}(\boldsymbol{X})] - \boldsymbol{E}[\boldsymbol{X}|\boldsymbol{X}+\boldsymbol{Y} > \boldsymbol{Q}_{\alpha}(\boldsymbol{X}+\boldsymbol{Y})] + \boldsymbol{E}[\boldsymbol{Y}|\boldsymbol{Y} > \boldsymbol{Q}_{\alpha}(\boldsymbol{Y})] \\
&\qquad - \boldsymbol{E}[\boldsymbol{Y}|\boldsymbol{X}+\boldsymbol{Y} > \boldsymbol{Q}_{\alpha}(\boldsymbol{X}+\boldsymbol{Y})],
\end{aligned}$$

which can seen by definition to be greater than zero since both expected value differences are nonnegative, proving subadditivity.

Another problem with VaR, or percentile funding, that is solved by CTE is what the author has termed the "Lognormal Paradox." As discussed in Smart (2008, 2012), with funding levels at or below the 84th percentile, for a common mean and standard deviation, a normal distribution will require more funding than a lognormal distribution even though the lognormal has a fatter right tail. This is contrary to common sense, which tells us that distributions with fatter right tails correspond to higher probabilities of risk events and therefore should require greater funding. See Figure 4 for a graphical comparison. As is evident from Figure 4, for percentiles between the 23rd and 84th percentiles, the normal distribution has higher percentile levels than the lognormal distribution. This occurs

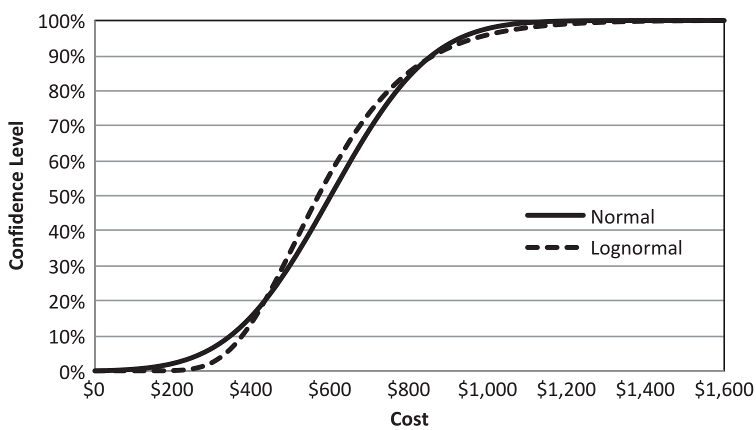

FIGURE 4 Comparison of a lognormal and normal distribution percentiles with common mean \$600 and common standard deviation \$200.

TABLE 5 Comparison of VaR and CTE

	Mean = \$600, Standard Deviation = \$200								
	$\alpha =$	50.0%	60.0%	70.0%	80.0%	90.0%	95.0%	99.0%	99.9%
VaR	Normal	\$600	\$651	\$705	\$768	\$856	\$929	\$1,065	\$1,218
	Lognormal	\$569	\$618	\$675	\$748	\$863	\$971	\$1,211	\$1,552
CTE	Normal	\$760	\$793	\$832	\$880	\$951	\$1,013	\$1,133	\$1,273
	Lognormal	\$753	\$793	\$842	\$908	\$1,016	\$1,120	\$1,359	\$1,704

despite the fact that their means and standard deviations are the same and the lognormal has fatter tails than the normal distribution, all else being equal. The cause of the paradox is that percentile funding does not take into account the full right tail, which is where the lognormal's extreme risk is located (even though the probability of that extreme risk is still small).

However CTE does not suffer this shortcoming, precisely because it takes the full right tail into account. See Table 5 for a comparison, where, for example, the percentiles for the lognormal and normal do not cross until the 90th percentile for VaR, while for CTE the lognormal is greater than the normal for all percentiles above the 60th percentile. Thus, CTE has another advantage, in that it is a more sensible policy.

Note that CTE funding requires additional money above and beyond strict percentile funding. This decreases as a percentage of the percentile as the percentile increases. For a lognormal funded at the 70th percentile, the additional funding needed to make up the expected shortfall is approximately 25% greater, while at the 80th percentile it is 21% more.

CTE is simple to calculate. When a lognormal or normal distribution is used to model total cost risk, as in the NASA/Air Force Cost Model (NAFCOM), there is a closed form equation that can be used to calculate CTE. These formulas, which can be implemented in a spreadsheet, are derived in Appendix B. And CTE can be easily calculated when Monte Carlo simulation is used to estimate cost risk. For example in a 10-trial Monte Carlo simulation of a normal distribution with mean equal to \$600 and standard deviation equal to \$200, whose trial values are shown in Table 6, the 70th percentile represents values above \$687.21, so to calculate $CTE_{0.70}$ we calculate the mean of the three values above the 70th percentile, namely the eighth, ninth, and tenth largest of the Monte Carlo trial values. Those values are \$732.19, \$755.82, and \$779.58, and their mean is equal to \$755.86. Incidentally,

TABLE 6 Monte Carlo trial values

1	379.69
2	450.73
3	451.91
4	504.46
5	548.09
6	661.94
7	687.21
8	732.19
9	755.82
10	779.58

because CTE is a mean, probability theory advises that its calculation should require, in most situations, fewer trials to accurately measure within a given error-bound range than percentiles, which require more trials for the same degree of accuracy.

Note that the mean calculated from the Monte Carlo trials, \$755.86, is still quite different from the exact value of \$832 calculated in Table 5. This inaccuracy results from the small number (ten) of samples used in this example. More trials would have resulted in a closer approximation.

CTE was introduced in the late 1990s and quickly became the preferred standard for setting liabilities for insurance settings. In Canada, the "actuarial Standards of Practice promulgate the use of the CTE whenever stochastic methods are used to set balance sheet liabilities" (Hancock and Manistre, 2005). It is also the basis for the Swiss Solvency Test (Filipovic and Kupper, 2007), which forms a major part of Swiss insurance policy. And the National Association of Insurance Commissioners recommends setting reserves using CTE (Lombardi, 2009).

The Practical Impact of Conditional Tail Expectation

The question still outstanding is whether or not the percentile funding policy being implemented by Government agencies will be effective in containing cost growth. Fewer missions overall should experience cost growth, but what about those that do? As we have shown percentile funding is not a true risk management policy, because additional funding, perhaps a significant amount, will be required much of the time, possibly 30% or more. It will likely be more, since even with 70th percentile funding, cost risk analyses typically explicitly exclude extreme events that occur from time to time, such as strikes, "acts of God," and other external factors beyond a project's control. However, should the overall amount needed above and beyond the 70th percentile be a relatively small amount, perhaps a percentile funding policy will help to stem cost growth. However, the historical record indicates this may be yet another pipe dream.

To gain an understanding of how much additional funding will be required in practice for percentile funding, it is useful to examine historical cost growth data. As discussed by Smart (2009), for a data set of cost growth for 112 recent NASA missions, some missions underran their estimates, others came in spot-on their budgets, and still others overran their budgets, often by large margins.

The minimum cost growth was −25.2% for Super Light Weight Tank, an upgrade for the space shuttle program from its more traditional aluminum structure to an aluminum-lithium composite. The negative number signifies an underrun of approximately 25%. (Contrary to popular belief, missions occasionally come in under budget.) For the current study, 14 such missions experienced underruns, which represented 12.5% of the missions studied.

Only two of the 112 missions hit their budget targets spot on. Nine of the missions were within 5% of the initial budget, and 19 were within 10% (either above or below), performance that is considered quite satisfactory.

The remaining missions' costs, the great majority of the 112 in fact, experienced cost growth in excess of 10% of the budgeted amount. Maximum cost growth among the missions studied was 385% for the Hubble Space Telescope and Space Telescope Assembly, which suffered from several sources of traditional cost growth, including funding constraints, launch vehicle delays, and underestimation of time and resources necessary to develop the requisite technology.

A range from −25% on the low side to over 350% on the high end is a wide range. The average (mean) cost growth for all missions was 53.0%, with median growth equal

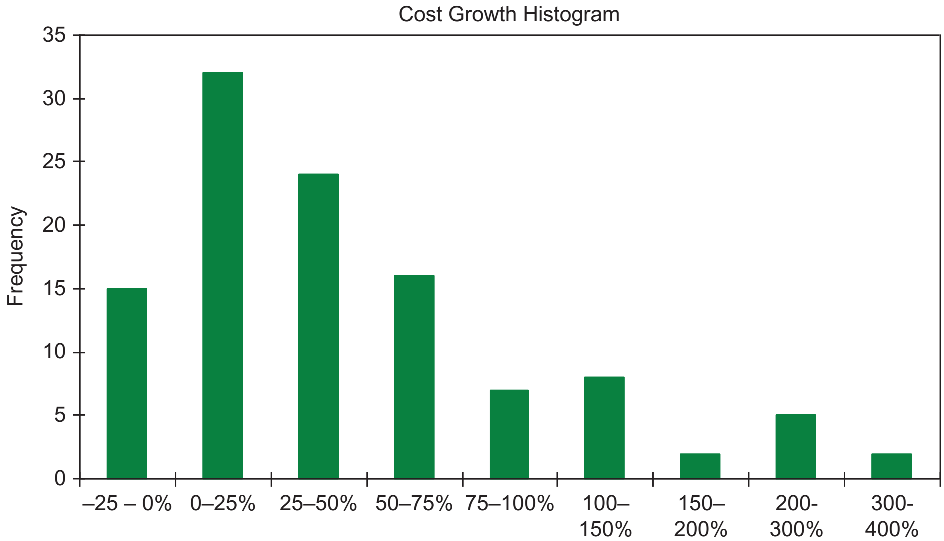

FIGURE 5 Graphical summary of NASA cost growth (color figure available online).

to 32.1%. The difference between the mean and median indicates a high degree of positive skew in the data, with most missions experiencing relatively small amounts of cost growth (half experienced growth less than 33%) and some, such as Hubble, experiencing extreme amounts of cost growth. Overall, 17 missions showed cost growth in excess of 100%, implying that costs more than doubled. While representing only about 15% of the 112 cost growth data points, we will see that growth of this severity, while not common, occurs often enough to offset any hoped-for portfolio effect. See Figure 5 for a graphical summary of these data.

The probability that an estimate will exceed a specified amount, such as \$100 million or \$150 million, is a measure of cost risk. Cost growth and cost risk are thus intrinsically related. Historical cost growth provides an excellent means for determining the overall level of risk for cost estimates. For example, if 95% of past programs have experienced less than 100% growth, we should expect that the ratio of actual cost to the initial estimate should be less than 100% with 95% confidence. Thus cost growth is the impact of cost risk in action. Because of uncertainty in historical data, cost models, program parameters, etc., the term “cost risk” is redundant since risk is inherent in cost estimates. Thus, characteristics of this cost growth data set determine characteristics seen in a cost risk distribution that is consistent with cost growth. In Smart (2009) it was shown that the cost growth data closely follow a lognormal distribution with a coefficient of variation equal to 100%. This implies a significant amount of cost risk that is much higher than that typically modeled by cost analysts. Note that this is a different conclusion than was reached in the author’s recent article in the *Journal of Cost Analysis and Parametrics* (2012) for a smaller data set of 40 missions.

Table 7 displays, for lognormal cost risk distributions of various levels of coefficient of variation (abbreviated CoV, it is the standard deviation divided by mean, expressed as a percentage), the additional amounts expected to be required if the budget is exceeded. This amount ranges from a low of around 10% of the budget if the latter is set at the 90th percentile for a distribution having CoV = 20%, to a high of about185% if the budget is set at the 30th percentile and CoV = 100%. Note that, for NASA, with its 70th percentile funding

TABLE 7 Additional funding needed if budget exceeded for various budget strategies and risk characteristics

Coefficient of Variation	Budget Set at 30th	40th	50th	60th	70th	80th	90th
20%	23.6%	20.5%	18.0%	15.9%	14.0%	12.2%	10.2%
30%	38.0%	32.7%	28.5%	25.0%	21.9%	19.0%	15.8%
40%	54.1%	46.1%	40.0%	34.9%	30.4%	26.2%	21.6%
50%	72.0%	60.9%	52.4%	45.5%	39.4%	33.7%	27.7%
60%	91.6%	76.8%	65.7%	56.7%	48.8%	41.5%	33.9%
70%	112.7%	93.8%	79.7%	68.3%	58.5%	49.5%	40.1%
80%	136.0%	112.0%	94.4%	80.5%	68.6%	57.6%	46.4%
90%	160.0%	131.0%	110.0%	93.0%	78.8%	65.9%	52.7%
100%	185.0%	151.0%	125.5%	105.8%	89.2%	74.2%	59.0%

policy and a CoV implied by the data to be 100%, missions that experience cost overruns beyond the 70th percentile, will require on average additional funding in the amount of 89% of the original budget. Here there be dragons indeed! This is a sobering amount, because 30% of the time, approximately 90% more money will be needed, even if the risk models being applied are calibrated to empirical cost growth experience.

Since 30% of the time, an additional 90% more funding will be required, the average project should expect to experience 27% growth ($.27 = 0.90 * .30$). While an improvement over the current 53% average growth, we can see that percentile funding will not be the hoped-for panacea, but only a band-aid where major surgery is required.

Summary

Current risk management policy for NASA and other agencies consists largely of setting reserves at a fixed percentile, popularly known outside the cost community as VaR. This policy has much in common with setting risk reserves in the banking industry. However, it ignores the tails of the risk distributions, ignorance of which is dangerous to the financial viability of the project. And funding to a percentile does not even provide a cushion for bad times; exceeding a percentile-funding level simply tells you that times are indeed bad. Percentile funding will not lead to an end to cost growth—empirical evidence suggests that a 70th percentile funding policy will result, on average, in a significant amount of cost growth. Furthermore, running to Congress and asking for more money when a fixed percentile is exceeded is not a risk-management policy, but rather a reflection of a lack of maturity and discipline required to fully implement sophisticated and meaningful risk management. Worst of all, percentile funding can result in a reverse portfolio effect, which means that funding an agency as a whole could be riskier than funding any single project! A better policy would be to use a risk measure such as CTE that takes into account the tails of the distribution. Such a policy will offer both a signal of a bad event (i.e., VaR is exceeded), as well as a cushion for the expected amount of money to guard against this event. CTE is a simple measure, represented by a single number just like percentile funding and can be easily explained to senior management and project managers, since it is simply the additional amount of money required to fund a project in case the percentile-funding budgeted is breached. It need not be significantly more expensive for the agency as a whole than current percentile-funding policies. Since it takes into account the full right tail of the distribution, a lower-level threshold such as the 60th or 70th percentile could be chosen for the trigger. In summary, a reserves strategy cannot stop with simply setting reserves at a relatively high percentile. Without a change in budgeting policy, government agencies should expect to incur much, much more spending on a regular basis.

References

Acerbi, C., & Tasche, D. (2002). On the Coherence of Expected Shortfall. *Journal of Banking and Finance*, 26, 1487–1503.

Anderson, T. P. (2004). The Trouble With Budgeting to the 80th Percentile. 72nd Military Operations Research Society Symposium, Monterey, CA, 22–24 June 2004.

Artzner, P., Delbaen, F., Eber, J., & Heath, D. (1999). Coherent Measures of Risk. *Mathematical Finance*, 9, 203–228.

Brown, A., & Einhorn, D. (2008). Private Profits and Socialized Risks. *GARP Risk Review*, July, 12.

Embrechts, P., McNeil, A., & Straumann, D. (2002). Correlation and Dependence in Risk Management: Properties and Pitfalls. In M.A.H. Dempster (Eds.), *Risk Management: Value at Risk and Beyond* (pp. 176–223). Cambridge: Cambridge University Press.

Embrechts, P., McNeil, A., & Frey, R. (2005). *Quantitative Risk Management*. Princeton: Princeton University Press.

Filipovic, D., & Kupper, M. (2007). On the Group Level Swiss Solvency Test. White paper. Retrieved from http://wws.mathematik.hu-berlin.de/~kupper/papers/GroupSST.pdf

Hancock, G., & Manistre, B. (2005). Variance of the CTE Estimator. *North American Actuarial Journal*, 9, 129–156.

Jorion, P. (2007). *Value at Risk: The New Benchmark for Financial Risk*, 3rd Edition. New York: McGraw-Hill.

Klugman, S.A., Panjer, H.H., & Willmot, G.E. (2008). *Loss Models: From Data to Decisions*, 3rd Ed. New York: John Wiley & Sons.

Lombardi, L. (2009). *Valuation of Life Insurance Liabilities: Establishing Reserves for Life Insurance Policies and Annuity Contracts*. Winsted, CT: ACTEX Publications.

Markowitz, H. (1959). *Portfolio Selection: Efficient Diversification of Investments*. Wiley, Yale University Press.

Smart, C.B. (2008, May). The Fractal Geometry of Cost Risk. Joint ISPA/SCEA Annual Conference, Noordwijk, the Netherlands.

Smart, C.B. (2009, June). The Portfolio Effect and the Free Lunch. Joint ISPA/SCEA Annual Conference, St. Louis, MO.

Smart, C.B., (2010, June). Here There Be Dragons: Incorporating the Right Tail in Risk Management. Joint ISPA/SCEA Annual Conference, San Diego, CA.

Smart, C.B., (2012). The Fractal Nature of Cost Risk: The Portfolio Effect, Power Laws, and Risk and Uncertainty Properties of Lognormal Distributions. *Journal of Cost Analysis and Parametrics*, 5(1), 5–24.

Taleb, N. (2007). *The Black Swan: The Impact of the Highly Improbable*. New York: Random House.

Triana, P. (2009). *Lecturing Birds on Flying: Can Mathematical Theories Destroy the Markets?*, New York: John Wiley & Sons.

Wirch, J. (1999). Raising Value at Risk. *North American Actuarial Journal*, 3, 106–115.

About the Author

Christian B. Smart, Ph.D., is the Director of Cost Estimating and Analysis for the Missile Defense Agency. In this capacity, he is responsible for overseeing all cost estimating activities developed and produced by the agency, and directs the work of 100 cost analysts. Prior to joining MDA, Dr. Smart worked as a senior parametric cost analyst and program manager with Science Applications International Corporation. An experienced estimator

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Appendix A

This appendix describes an example of a case of superadditivity in a percentile-funding strategy, namely the possibility that cost risk of a portfolio may exceed the combined cost risk of the projects in the portfolio. Consider two independent random variables $\boldsymbol{X}$, $\boldsymbol{Y}$, each of which follows a one-parameter Pareto distribution, whose distribution function is defined as

$$\boldsymbol{F_{Par}(x) = Pr\{Par \le x\} = 1 - \left(\frac{\theta}{x}\right)^{\tau}} \textbf{ for } \boldsymbol{x \ge \theta}$$

$$\boldsymbol{= 0} \textbf{ for } \boldsymbol{x < \theta}.$$

This distribution is referred to as a one-parameter Pareto, because the parameter $\boldsymbol{\theta}$ is merely a barrier designator that establishes the domain of the distribution, unlike the parameters of the two-parameter Pareto that is described in the Introduction (Klugman et al., 2008), where both parameters actively describe the analytic behavior of the distribution function.

For purposes of this example, suppose that each of the random variables $\boldsymbol{X}$ and $\boldsymbol{Y}$ has the specific parameters $\boldsymbol{\tau = \frac{1}{2}}$ and $\boldsymbol{\theta = 1}$, which means that that $\boldsymbol{F_{Par}(y) = 1 - y^{-1/2}}$ for $\boldsymbol{x \ge 1}$ and $\boldsymbol{F_{Par}(x) = 0}$ otherwise and $\boldsymbol{F_{Par}(y) = 1 - y^{-1/2}}$ for $\boldsymbol{y \ge 1}$ and $\boldsymbol{F_{Par}(y) = 0}$ otherwise, respectively. Then, to determine the probability that the sum of the two Pareto variables $\boldsymbol{X}$ and $\boldsymbol{Y}$ is less than or equal to some value z, convolution of probability distributions is applied. The solution space $\boldsymbol{X + Y} \le z$, bounded on the upper right by the straight line $\boldsymbol{Y} = z - \boldsymbol{X}$, is represented in Figure A.1, where for each value of $\boldsymbol{X}$, $\boldsymbol{Y}$ ranges in value from 1 to $z - \boldsymbol{X}$ and, to complete the triangle, $\boldsymbol{X}$ sweeps from 1 to $z - \boldsymbol{1}$ (since $\boldsymbol{Y}$ is at least 1). The straight line $\boldsymbol{Y} = z - \boldsymbol{X}$ is the boundary between the regions where $\boldsymbol{X + Y} < z$ and $\boldsymbol{X + Y} > z$. Therefore $\boldsymbol{Pr(X + Y} \le z)$ is the probability of the triangular region below and to the left

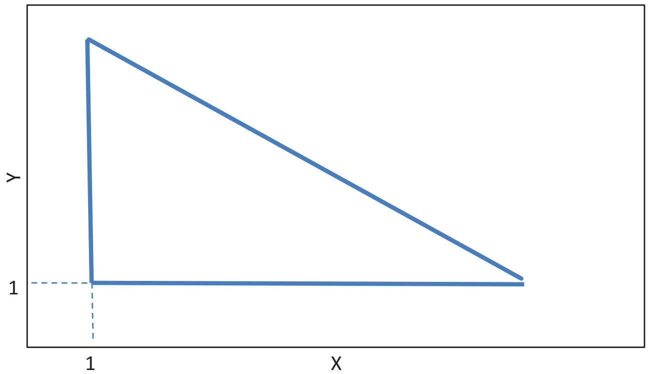

FIGURE A.1 Solution space for $\Pr(X + Y \leq z)$ (color figure available online).

of the line, bounded by its base and height intersecting at the point *(1,1)*. The probability density functions of X and Y are the derivatives of their respective distribution functions, namely $f(x) = ½\, x^{-3/2}$ and $g(y) = ½ y^{-3/2}$.

Denoting the distribution function of the sum $X + Y$ as G, the probability of interest is calculated as the double integral over the triangular region in Figure A.1, namely,

$$
\begin{aligned}
G(z) = Pr(X + Y \leq z) &= \int_1^{z-1}\int_1^{z-x} f(x)g(y)dydx \\
&= \int_1^{z-1} \frac{1}{2}x^{-3/2}\left[\int_1^{z-x}\frac{1}{2}y^{-3/2}dy\right]dx \\
&= \int_1^{z-1} \frac{1}{2}x^{-3/2}\left[-y^{-1/2}\right]_1^{z-x}dx = \int_1^{z-1}\frac{1}{2}x^{-3/2}(1-(z-x)^{-1/2})dx \\
&= \int_1^{z-1}\frac{1}{2}x^{-\frac{3}{2}}dx - \frac{1}{2}\int_1^{z-1} x^{-\frac{3}{2}}(z-x)^{-\frac{1}{2}}dx \\
&= \left[-x^{-1/2}\right]_1^{z-1} - \frac{1}{2}\int_1^{z-1} x^{-3/2}(z-x)^{-1/2}dx \\
&= 1 - \frac{1}{\sqrt{z-1}} - \frac{1}{2}\int_1^{z-1} x^{-3/2}(z-x)^{-1/2}dx.
\end{aligned}
$$

In order to calculate the last integral with respect to x, set $u = \sqrt{x}$ so that $du = \frac{1}{2\sqrt{x}}dx$, which in turn means that $2\sqrt{x}du = dx$. Substituting u for x then yields $2udu = dx$. Therefore (ignoring integration limits for now),

$$1 - \frac{1}{\sqrt{z-1}} - \frac{1}{2}\int x^{-3/2}(z-x)^{-1/2}dx = 1 - \frac{1}{\sqrt{z-1}} - \frac{1}{2}\int u^{-3}(z-u^2)^{-1/2}2udu,$$

which simplifies to $1 - \frac{1}{\sqrt{z-1}} - \int u^{-2}(z-u^2)^{-1/2}du$.

A further change of variable is needed at this point, in particular a trigonometric substitution of $\boldsymbol{s}$ for $\boldsymbol{u}$ as follows, continuing to ignore the integration limits: set $\boldsymbol{u} = \sqrt{z}\boldsymbol{sin}(s)$, and, thus, $\boldsymbol{du} = \sqrt{z}\,\boldsymbol{cos}(s)\boldsymbol{ds}$. Then,

$$\sqrt{z - u^2} = \sqrt{z - z\sin^2(s)} = \sqrt{z}\cos(s)$$

and $s = \sin^{-1}\left(\frac{u}{\sqrt{z}}\right)$. Therefore, the integrand above becomes

$$\frac{1}{u^2\sqrt{z-u^2}}du = \frac{1}{z\sin^2(s)\sqrt{z}\cos(s)}\sqrt{z}\cos(s)ds = \frac{1}{z\sin^2(s)}ds = \frac{\csc^2(s)}{z}ds$$

And, hence, the indefinite integral for $\boldsymbol{G(z) = Pr(X+Y \leq z)}$ simplifies to

$$\begin{aligned} 1 - \frac{1}{\sqrt{z-1}} - \int \frac{1}{u^2\sqrt{z-u^2}}du &= 1 - \frac{1}{\sqrt{z-1}} - \int \frac{\csc^2(s)}{z}ds \\ &= 1 - \frac{1}{\sqrt{z-1}} + \left[\frac{\cot(s)}{z}\right] + c, \end{aligned}$$

where $\boldsymbol{c}$ is an arbitrary constant to be later superseded by re-inserting the integration limits.

Note that since $s = \sin^{-1}\left(\frac{u}{\sqrt{z}}\right)$,

$$\begin{aligned} \cot(s) = \frac{\cos(s)}{\sin(s)} = \frac{\cos\left(\sin^{-1}\left(\frac{u}{\sqrt{z}}\right)\right)}{\frac{u}{\sqrt{z}}} &= \frac{\sqrt{\cos^2\left(\sin^{-1}\left(\frac{u}{\sqrt{z}}\right)\right)}}{\frac{u}{\sqrt{z}}} \\ &= \frac{\sqrt{1 - \sin^2\left(\sin^{-1}\left(\frac{u}{\sqrt{z}}\right)\right)}}{\frac{u}{\sqrt{z}}} = \frac{\sqrt{1 - \frac{u^2}{z}}}{\frac{u}{\sqrt{z}}}, \end{aligned}$$

From which it follows that $1 - \frac{1}{\sqrt{z-1}} + \left[\frac{\cot(s)}{z}\right] = 1 - \frac{1}{\sqrt{z-1}} + \frac{\sqrt{1-\frac{u^2}{z}}}{z\frac{u}{\sqrt{z}}} = 1 - \frac{1}{\sqrt{z-1}} + \frac{\sqrt{1-\frac{u^2}{z}}}{u\sqrt{z}}$.

Now, since $\boldsymbol{u} = \sqrt{x}$, finally putting the integration limits back in yields

$$\begin{aligned}G(z) = Pr(X+Y\le z) &= 1-\frac{1}{\sqrt{z-1}}-\frac{1}{2}\int_1^{z-1} x^{-3/2}(z-x)^{-1/2}dx\\ &= 1-\frac{1}{\sqrt{z-1}}+\left[\frac{\sqrt{1-\frac{x}{z}}}{\sqrt{xz}}\right]_1^{z-1}\\ &= 1-\frac{1}{\sqrt{z-1}}+\left[\frac{\sqrt{z-x}}{z\sqrt{x}}\right]_1^{z-1}\\ &= 1-\frac{1}{\sqrt{z-1}}+\frac{1}{z\sqrt{z-1}}-\frac{\sqrt{z-1}}{z}\\ &= 1-\frac{2\sqrt{z-1}}{z}.\end{aligned}$$

Returning now to the superadditivity issue, note that

$$Pr(2X\le z) = Pr\left(X\le\frac{z}{2}\right) = 1-\left(\frac{z}{2}\right)^{-\frac{1}{2}} > 1-\frac{2\sqrt{z-1}}{z} = Pr(X+Y\le z),$$

if ≥ 2. But $z<2$ is not a possibility because $\theta=1$ (in the one-parameter Pareto distribution we are working with) implies that $X\ge 1$, so that $2X\ge 2$. So, if $2X\le z$, z must be at least 2.

In this case, $VaR_\alpha(X) = F^{-1}(\alpha)$, as discussed in the Introduction. Since

$$F\left(\frac{z}{2}\right) = Pr\left(X\le\frac{z}{2}\right) = Pr(2X\le z)\ge Pr(X+Y\le z) = G(z)$$

implies that, for any number $z\ge 2$, the probability that $2X$ does not exceed z is greater than the probability that $X+Y$ does not exceed z.

As an example of a possible value of z, consider the $(1-\alpha)$th percentile of $2X$, namely, the value of z for which $Pr(2X\le z) = 1-\alpha$. The probability that $X+Y$ is less than or equal to that value of z must, therefore, be smaller than $1-\alpha$, and this ensures that the $(1-\alpha)$th percentile of $X+Y$ must be larger than z, that is, $VaR_\alpha(X+Y) > VaR_\alpha(2X)$.

Now, recalling that $VaR_\alpha(X) = F^{-1}(\alpha)$, where $F(x) = 1-x^{-1/2}$ for our example, it follows that $VaR_\alpha(X) = (1-\alpha)^{-2}$ by solving the equation $1-\alpha = 1-VaR_\alpha(X)^{-1/2}$, where $x = VaR_\alpha(X)$ is the $(1-\alpha)$th percentile of X. Additionally, $VaR_\alpha(2X) = 2(1-\alpha)^{-2}$ follows directly from the positive homogeneity of VaR.

Because X and Y have the same (one-parameter Pareto) distribution, the fact that $VaR_\alpha(X) = (1-\alpha)^{-2}$ implies also that $VaR_\alpha(Y) = (1-\alpha)^{-2}$. A consequence of the fact that $VaR_\alpha(2X) = 2(1-\alpha)^{-2}$ is then that

$$VaR_\alpha(X+Y) > VaR_\alpha(2X) = VaR_\alpha(X)+VaR_\alpha(Y),$$

one instance of which asserts, for $\alpha = 0.20$, that the 80th percentile of funding for the portfolio consisting of projects X and Y exceeds the sum of the individual 80th percentiles of funding for projects X and Y considered separately.

In conclusion, one contrary example is sufficient to disprove a proposed general principle, and this example establishes that, for percentile funding, not only is there no guaranteed portfolio effect and that there may in fact be a negative portfolio effect. If we fund individual projects at a specified percentile of the cost risk S-curve, it may be that the total portfolio funding, namely the sum of the budgets of the individual projects, may not be sufficient to fund the portfolio at even that same level of confidence (not yet mentioning a higher level of confidence). Farther down the line, this may mean that greater risk reserves will be needed to achieve the desired confidence levels of the individual projects. This possibility is clearly undesirable for a risk management policy and calls into question the use of percentile funding when setting risk reserves for government projects.

Appendix B

Note that for a normal distribution, $VaR_\alpha(X) = \mu + \sigma\Phi^{-1}(\alpha)$, and that

$$CTE_\alpha(X) = \mu + \sigma E\left[\frac{X-\mu}{\sigma}\middle|\frac{X-\mu}{\sigma} \geq Q_\alpha\left(\frac{X-\mu}{\sigma}\right)\right].$$

Note that

$$\begin{aligned}
E\left[\frac{X-\mu}{\sigma}\middle|\frac{X-\mu}{\sigma} \geq Q_\alpha\left(\frac{X-\mu}{\sigma}\right)\right] &= \frac{1}{1-\alpha}\int_{\Phi^{-1}(\alpha)}^{\infty} x\phi(x)dx \\
&= \frac{1}{1-\alpha}\left[-\phi(x)\right]_{\Phi^{-1}(\alpha)}^{\infty} \\
&= \frac{\phi\left(\phi^{-1}(\alpha)\right)}{1-\alpha}
\end{aligned}$$

where ϕ represents the standard normal density function and ϕ^{-1} represents the inverse of the cumulative standard normal distribution. Therefore, for a normal distribution,

$$CTE_\alpha(X) = \mu + \sigma\frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}.$$

For a lognormal distribution,

$$CTE_\alpha = VaR_\alpha + \frac{E[X] - E[XVaR_\alpha]}{1-\alpha}.$$

Note that $E[XVaR_\alpha] = \int_0^{VaR_\alpha} \frac{1}{\sqrt{2\pi}\sigma} exp\left(-\frac{1}{2}\left(\frac{lny-\mu}{\sigma}\right)^2\right)dy + VaR_\alpha(1-\alpha)$, and setting $z = \frac{lny-\mu-\sigma^2}{\sigma}$, the integral simplifies to

$$exp\left(\mu+\frac{\sigma^2}{2}\right)\int_{-\infty}^{\frac{lnVaR_\alpha-\mu-\sigma^2}{\sigma}} \frac{1}{\sqrt{2\pi}} exp\left(-\frac{1}{2}z^2\right)dz = E[X]\left[\Phi\left(\frac{\ln VaR_\alpha-\mu-\sigma^2}{\sigma}\right)\right].$$

Thus, the CTE for the lognormal distribution can be written as

$$VaR_\alpha + \frac{E[X] - E[X]\left[\Phi\left(\frac{lnVaR_\alpha-\mu-\sigma^2}{\sigma}\right)\right] - VaR_\alpha(1-\alpha)}{1-\alpha}$$

$$= \frac{\boldsymbol{E[X]}\left[\mathbf{1} - \boldsymbol{\Phi}\left(\frac{\boldsymbol{lnVaR_{\alpha} - \mu - \sigma^2}}{\boldsymbol{\sigma}}\right)\right]}{\mathbf{1} - \boldsymbol{\alpha}},$$

where Φ is the cumulative standard normal distribution function.

Appendix C: Acronyms

CoV	Coefficent of Variation
CTE	Conditional Tail Expectation
DoD	Department of Defense
FFP	Firm Fixed Price
NAFCOM	NASA Air Force Cost Model
NASA	National Aeronautics and Space Administration
TVaR	Tail Value at Risk
VaR	Value at Risk