# Robust Default Correlation for Cost Risk Analysis

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# Introduction

- **Correlation is an important consideration in cost risk analysis**
- **When correlation is ignored, you are making the de facto assumption that all risks are independent**
  - **"Even when you choose not to decide, you still have made a choice" (Rush, *Free Will*)**
- **Assuming no correlation results in a vast understatement of risk**
- **In 1996, Don Mackenzie wrote that "One of the more difficult chores in cost risk analysis is establishing appropriate levels of correlation… " (Mackenzie 1996)**
  - **Seventeen years later, this is still true**
- **This presentation is an attempt at making forward progress on this issue**

# Definitions

- **Consider two random variables, *X* and *Y*.**
- **The mean of X, E(X), is denoted by $\mu_x$, and similarly, the mean of Y, E(Y), is denoted by $\mu_y$**
- **The variance of X, Var(X), is denoted by $\sigma_X^2$, and similarly, the variance of Y, Var(Y), is denoted by $\sigma_Y^2$**
- **The variance of X and Y are equal to:**

$$Var(X) = Cov(X,X) = E(X^2) - [E(X)]^2$$

$$Var(Y) = Cov(Y,Y) = E(Y^2) - [E(Y)]^2$$

- **Correlation, denoted by the Greek letter *r* ("rho"), is defined by**

$$\rho_{XY} = Corr(X,Y) = \frac{cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{E(XY) - E(X)E(Y)}{\sqrt{Var(X)Var(Y)}} = \frac{E(XY) - \mu_X \mu_Y}{\sigma_X \sigma_Y}$$

# Total System Mean and Variance

- **For *n* WBS elements, the mean and the variance of the total cost are defined by:**

$$E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} \mu_i$$

$$Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=}^{n} \sigma_i^2 + 2\sum_{j=2}^{n}\sum_{i=1}^{j-1} \rho_{ij}\sigma_i\sigma_j$$

# Total Variance with Level Correlation

- **Suppose (for simplicity)**
  - **There are *n* WBS Elements** $C_1, C_2, \ldots, C_n$
  - **Each** $Var(C_i) = \sigma^2$
  - **Each** $Corr(C_i, C_j) = \rho < 1$
  - **Total Cost** $C = \sum_{k=1}^{n} C_i$

$$Var(C) = \sum_{k=1}^{n} Var(C_i) + 2\rho \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sqrt{Var(C_i)\ Var(C_j)}$$

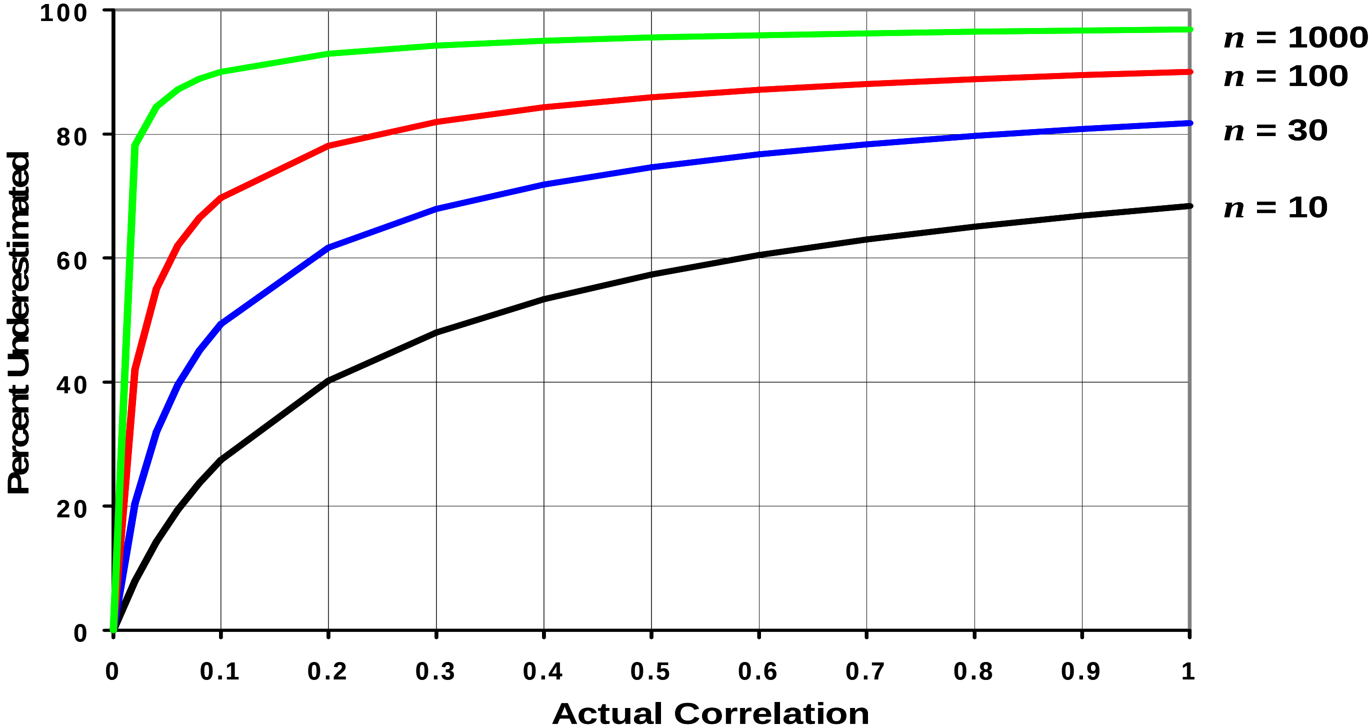
$$= n\sigma^2 + n(n-1)\rho\sigma^2$$

$$= n\sigma^2(1+(n-1)\rho)$$

| Correlation | 0 | ρ | 1 |
|---|---|---|---|
| $Var(C)$ | $n\sigma^2$ | $n\sigma^2(1+(n-1)\rho)$ | $n^2\sigma^2$ |

# Impact of Assuming Independence

- **For a 100 element WBS assuming independence among all WBS elements when the true underlying correlation is equal to 20% results in an underestimate of total system standard deviation equal to 80%!**




**Source: "Why Correlation Matters in Cost Estimating," Advanced Training Session, 32nd Annual DOD Cost Analysis Symposium, Williamsburg, VA, 1999.**

# Example of Impact

- **As an example, consider a system with 10 subsystems, each with mean equal to \$10 million and standard deviation equal to \$3 million**
- **For 100 elements and 1,000 elements, assuming correlation is zero when it is actually 20% results in underestimating the 80th percentile by 8-10%, and if the correlation is 60%, the 80th percentile is underestimated by 15-17%**

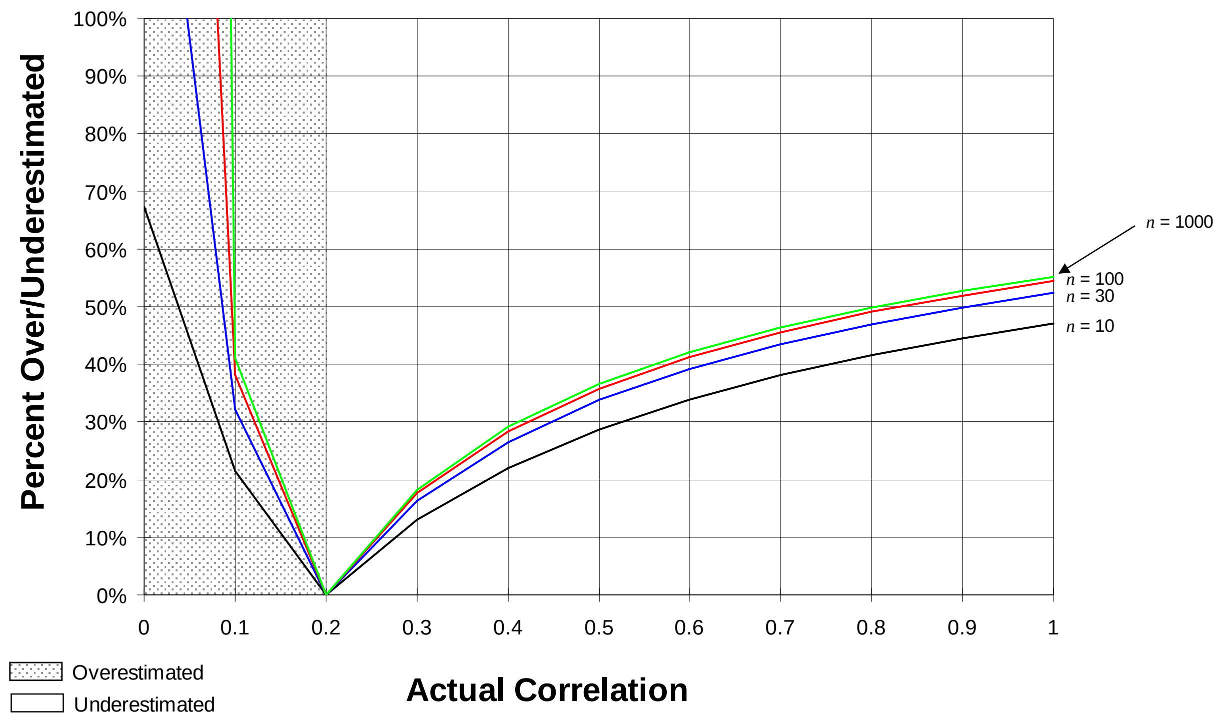
| | 80% Confidence Level (TY$, Millions) | | |
|---|---|---|---|
| Number of WBS Elements | Independence | 20% Correlation | 60% Correlation |
| 10 | $108 | $113 | $119 |
| 100 | $1,025 | $1,111 | $1,182 |
| 1,000 | $10,080 | $11,092 | $11,815 |

# Default Correlation

- **Notice in the graph on the previous chart there is an apparent "knee" in the curve around 20%**
  - **Above 20% correlation the consequence of assuming less correlation begins to dwindle**
  - **This graph is the basis for assuming 20-30% for default correlation for elements between which there is no functional correlation**
  - **Book (Book 1999) recommends 20% as a default correlation value because of this**
- **However, the graph does not tell us how much the total standard deviation is underestimated because correlation is assumed to be 20%, but is actually 60%, for example**

# Underestimating Correlation with the Default 20%

- **For a 100-element WBS, if the correlation is assumed to be 20% but is actually 60%, the total standard deviation is underestimated by 40%**




Source: "Why Correlation Matters in Cost Estimating," Advanced Training Session, 32nd Annual DOD Cost Analysis Symposium, Williamsburg, VA, 1999.

# Robust Approach

- **A more robust approach to assigning correlations would be to use the value that results in the least amount of error in the variance**
  - **It is robust in the sense that without solid evidence to assign a correlation value, it minimizes the amount by which the total standard is misestimated due to the correlation assumption**
  - **This robust default measure of correlation would be a value for correlation that would minimize the error when the assumed correlation differs from the actual underlying correlation**

# Absolute Error

- **We are interested in the absolute value of the error, since if we consider negative and positive values, they may offset each other**
- **Let ε denote the error, then we are interested in |ε|, where |ε| is defined by**

$$|\varepsilon| = \begin{cases} \varepsilon \text{ if } \varepsilon > 0 \\ -\varepsilon \text{ if } \varepsilon < 0 \end{cases}$$

# Expected Value of Absolute Error

- **If we assume that the prior distribution of correlation on the interval (0,1) is uniform, then the expected value of the absolute error |ε| of the variance as a function of the assumed correlation is defined by**

$$\int_0^1 |\varepsilon|\, f(\rho)\,d\rho = \int_0^1 |\varepsilon|\,d\rho$$

**since** $f(\rho) = 1$

- **Thus the approach is to find the value of $\rho$ that minimizes the expected (absolute) error**
- **This equation provides the expected error as a function of $\rho$, and then we minimize this function with respect to $\rho$ using techniques from elementary Calculus**

# What is the Error?

- **Now that we have determined how to determine the minimum error, we need to figure out what to minimize**
- **We present several different choices, calculate the results, and provide pros and cons for each**

# Case 1: Percentage Error (of Actual)

- **Denote the assumed correlation by $\rho_1$ and the actual correlation by $\rho_2$**
- **In this first case, we consider the metric that Book (Book, 1999) looked at when measuring over- and under-estimation of correlation, which is to consider the percentage error in variance as a percentage of the actual correlation**

$$\varepsilon = \frac{\sqrt{n}\sigma\sqrt{1+(n-1)\rho_2} - \sqrt{n}\sigma\sqrt{1+(n-1)\rho_1}}{\sqrt{n}\sigma\sqrt{1+(n-1)\rho_2}} = \frac{\sqrt{1+(n-1)\rho_2} - \sqrt{1+(n-1)\rho_1}}{\sqrt{1+(n-1)\rho_2}}$$

# Case 1: Calculating Expected Absolute Error (1 of 2)

- **The expected absolute error is calculated* as**

$$\int_0^{\rho_1} \frac{\sqrt{1+(n-1)\rho_1}-\sqrt{1+(n-1)\rho_2}}{\sqrt{1+(n-1)\rho_2}} d\rho_2 + \int_{\rho_1}^{1} \frac{\sqrt{1+(n-1)\rho_2}-\sqrt{1+(n-1)\rho_1}}{\sqrt{1+(n-1)\rho_2}} d\rho_2$$

$$= 2\rho_1 + \frac{4}{n-1} - \frac{2}{n-1}\sqrt{1+(n-1)\rho_1}\left(1+\sqrt{n}\right)+1$$

- **This is a function of the number of WBS elements (*n*) and the assumed correlation ($\rho_1$)**
- **Minimizing this with respect to $\rho_1$ we find that**

$$\rho_1 = \frac{\left(1+\sqrt{n}\right)^2 - 4}{4(n-1)}$$

***See the paper for detailed calculations**

# Case 1: Calculating Expected Absolute Error (2 of 2)

- **The limit of this minimum as $n \to \infty$ is 25%**
- **This is close to the 20% default value advocated by Book (Book, 1999)**
- **However, the total error is minimized by this value because of the large penalty assigned when overestimating actual correlations near zero**
- **For example, let *n= 100* and assume the correlation is 40%. The absolute percentage error when the actual correlation is equal to zero is 537%, while the absolute percentage error when the actual correlation is equal to 80% is only 29%**
- **The penalty should not differ greatly whether you are overestimating or underestimating**
  - **An easy way to overcome this issue is to examine the percent error as a function of the assumed correlation, which is considered in Case 2**

# Case 2: Percentage Error (of Assumed) (1 of 2)

- **This case is similar to Case 1, only the denominator is different**

$$\varepsilon = \frac{\sqrt{n}\sigma\sqrt{1+(n-1)\rho_2} - \sqrt{n}\sigma\sqrt{1+(n-1)\rho_1}}{\sqrt{n}\sigma\sqrt{1+(n-1)\rho_1}} = \frac{\sqrt{1+(n-1)\rho_2} - \sqrt{1+(n-1)\rho_1}}{\sqrt{1+(n-1)\rho_1}}$$

- **In this case,**

$$E(|\varepsilon|) = 2\rho_1 - \frac{4}{3(n-1)}\left(1+(n-1)\rho_1\right) + \frac{2}{3(n-1)}\left(1+n^{\frac{3}{2}}\right)\left(1+(n-1)\rho_1\right)^{-\frac{1}{2}} - 1$$

# Case 2: Percentage Error (of Assumed) (2 of 2)

- **The value of $\rho_1$ that minimizes the expected (absolute) error is**

$$\rho_1 = \frac{\left(\frac{1+n^{\frac{3}{2}}}{2}\right)^{\frac{2}{3}} - 1}{n-1}$$

- **The limit of $\rho_1$ as $n \to \infty$ is**

$$\left(\frac{1}{2}\right)^{\frac{2}{3}} \approx 63\%$$

# Impact of Case 2

- **The single recommended value from this approach is 63%**
  - **This is much larger than the 25% value using the other approach, or the 20% rule of thumb widely used in practice**

- **The impact on standard deviation in increasing default correlation from 20% to 63% will result in a significant increase in standard deviation**

| *n* | *% Increase in* $\sigma$ |
|---|---|
| 10 | 54.3% |
| 30 | 68.3% |
| 100 | 74.5% |
| 1,000 | 77.2% |
| 10,000 | 77.5% |

# Case 3: Total Absolute Difference

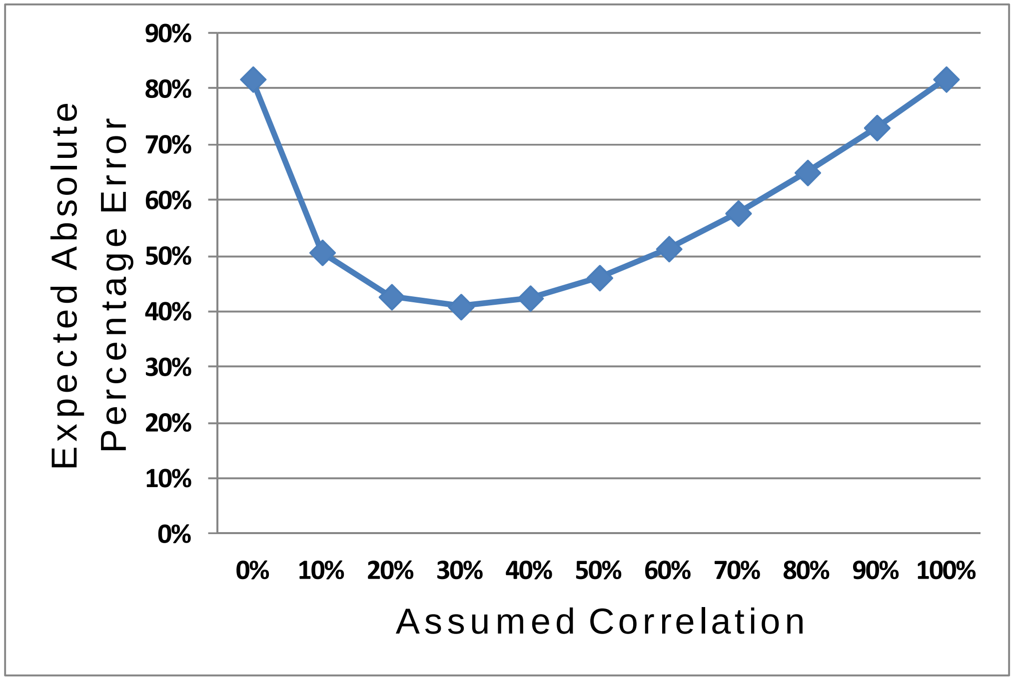
- **The absolute difference could also be considered as a metric**

$$\sqrt{n}\sigma\sqrt{1+(n-1)\rho_2} - \sqrt{n}\sigma\sqrt{1+(n-1)\rho_1}$$

- **In this case, the absolute expected value of the error occurs when $\rho_1$ = 50%**

# Case 4: Case 1 with Truncated Limits (1 of 2)

- **If we consider the first case, much of the reason why the minimum is so low compared to the other cases is the error when the actual correlation is close to 0%**
  - **We know that in most case the correlation is not 0%, and we know that it is not 100%**
- **Absolute percentage error for variance as a percent of the actual correlation for 100 WBS elements:**



# Case 4: Case 1 with Truncated Limits (1 of 2)

- **If we truncate the actual correlation to be uniform in the interval (0.1,0.9) then the expected value of the absolute percent error is minimized when**

$$\rho_1 = \frac{\left( (0.1n+0.9)^{\frac{1}{2}} + (0.9n+.1)^{\frac{1}{2}} \right)^2 - 4}{4(n-1)}$$

- **The limit of this as $n \to \infty$ is 40%**

# Summary of the Four Cases

- **All four cases minimize the expected value of the absolute error in the variance, but use different metrics for measuring error**
- **Case 1:**
  - **Error is measured as a percentage of the variance that results from the actual correlation, result in the limit is 25%**
- **Case 2:**
  - **Error is measured as a percentage of the variance that results from the assumed correlation, result in the limit is 63%**
- **Case 3:**
  - **Error is measured as total difference in variances, result is 50%**
- **Case 4:**
  - **Error is measured as a percentage of the variance that results from the actual correlation, with the correlation range limited to 10-90%; result is 40%**

# Recommendation

- **I recommend a percentage difference approach**
  - **Knowing that the difference between the estimated total standard deviation and the actual total standard deviation is $100 million doesn't tell you much, since it could be large if the standard deviation is $100 million, or relatively small if the total standard deviation is $1 billion**
- **Calculating the error based as a percentage of the assumed correlation is logical**
  - **The issue with looking at the error relative to the actual correlation is that we don't know the actual correlation - we only know the assumed correlation.**
  - **The same is true for CER residuals**
    - **For the Minimum Unbiased Percent Error (MUPE) and the Zero bias Minimum Percent Error (ZMPE) CER methods look at the percentage error from the estimate, not from the actual**
    - **We should use the same metric in looking at correlation**
- **Bottom line: I recommend using a default value for correlation that is equal to 63%**

# Empirical Evidence for Correlation

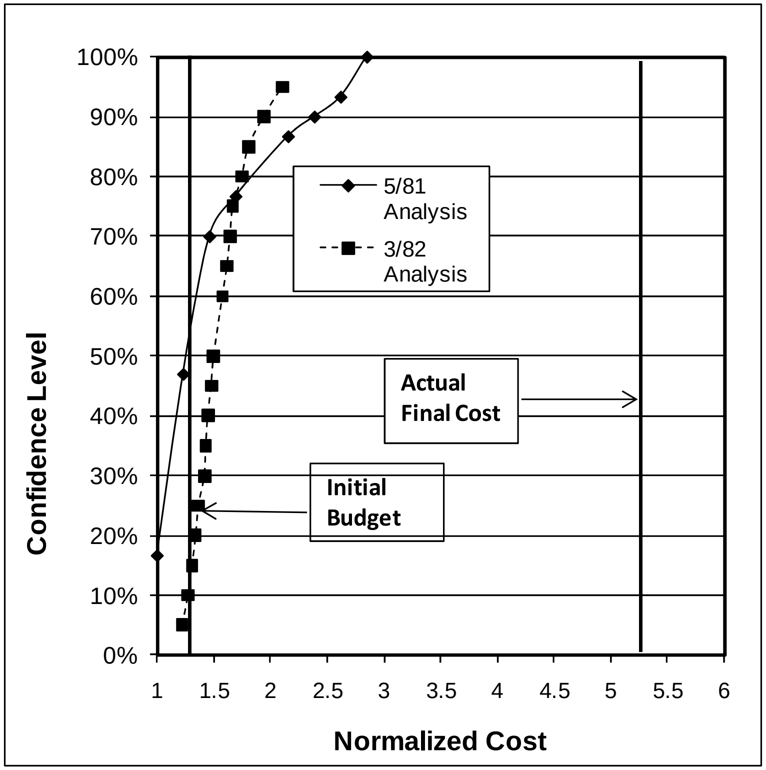
- **There is some limited empirical evidence on correlation for spacecraft**
- **This ranges from 16-40% at the subsystem level**
  - **Smart calculated an average correlation in the range 16-20% for NASA/Air Force Cost Model hardware subsystems (Smart, 2004)**
  - **Covert and Anderson calculated an average correlation equal to 16.8% for Unmanned Spacecraft Cost Model subsystems (Covert and Anderson, 2005)**
  - **Mackenzie and Addison reported correlations in the range 20-40% for average unit cost of subsystems NRO data (Mackenzie and Addison, 2000)**
- **However, this evidence is only for one commodity**

# Summary (1 of 2)

- **20% is often the default value when there is no information to provide informed input**
  - **This level is too low**
- **Using a more robust approach, we have shown that default values in the range 40-63% are more reasonable**
  - **I recommend 63% as a default value**
- **Only downside is potential for overestimation**
  - **However as a profession we do not have a reputation for overestimation**
  - **Increasing default correlation value may help counter this**

# Summary (2 of 2)

- **Example of underestimation of risk**
  - **For a risk analysis conducted for the Tethered Satellite System, the actual cost was more than double the 95th percentile of the original cost risk analysis**



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