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Mathematical Lessons Learned from a Year's Worth of ICEs

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Scitor Corporation

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Correlation

The Problem

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- **Needed a Monte Carlo simulation tool for a cost estimate conducted on a "stand-alone" system**
  - Did not have access to Monte Carlo SW package
- **Our team created a routine in Excel with Mathematical equations and Visual Basic Code**
- **Used @Risk to validate results**
  - Could have used Crystal Ball, packages are similar
- **Custom developed package appeared to work fine *except for the application of Correlation***
  - Accounting for an acceptable amount of variance due to the "randomness" of Monte Carlo simulation
  - Problem was using Excel's "Correl" function (Pearson) vs. @Risk's method (Spearman) to measure correlation
    - Was not an issue with the results but rather the validation method!

History of Correlation

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1888

Sir Francis Galton first introduced the concept of correlation through examining forearm and height measurements

1904

Charles Spearman developed rank correlation, a non-parametric version of the conventional Pearson correlation

2011

Albuquerque, New Mexico- SCEA

< 65 MYA

Dinosaurs ruled the Earth

1896

Karl Pearson introduces "Pearson's product-moment correlation coefficient"

1930s-1940s

WWII

Monte Carlo Simulation

*Photos from Wikipedia

Background

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- **What is correlation?**
  - A causal, complementary, parallel, or reciprocal relationship, especially a structural, functional, or qualitative correspondence between two comparable entities*
    - A zero correlation indicates that there is no relationship between the variables
    - A correlation of –1 indicates a perfect negative correlation, meaning that as one variable goes up, the other goes down
    - A correlation of +1 indicates a perfect positive correlation, meaning that both variables move in the same direction together
- **Correlation Coefficient Calculations:**
  - Pearson
    - In statistics, the **Pearson product-moment correlation coefficient** is a measure of the linear dependence between two variables *X* and *Y*
  - Spearman
    - **Spearman's Rank Correlation Coefficient** measures the extent to which, as one variable increases, the other variable tends to increase, without requiring that increase to be represented by a linear relationship. It is commonly used either to reduce the amount of calculation or to make the coefficient less sensitive to non-normality in distributions

*www.thefreedictionary.com

Pearson vs. Spearman in @Risk

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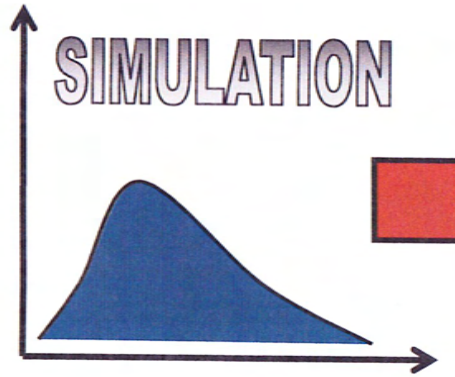
- **Example of the difference of Pearson and Spearman in @Risk**

DATASET

	SE Hrs	PM Hrs
Contract 1	1000	500
Contract 2	2000	650
Contract 3	2500	100
Contract 4	3000	900
Contract 5	3500	300
Contract 6	3800	1050
Contract 7	4000	450
Contract 8	5000	550
Contract 9	8000	800
Contract 10	10000	5000



CORRELATION OUTPUTS

Simulation Results	Spearman Input	Output Pearson	Output Spearman
Correlation SE-PM	0.442	0.227	0.440

Calculated correlation of 10,000 SE – PM trials to see if outputs matched the inputs

- Spearman holds
- Pearson does not

Simulation Results	Pearson Input	Output Pearson	Output Spearman
Correlation SE-PM	0.753	0.160	0.738

Pearson	0.753102
Spearman	0.442424

CALCULATIONS

SE Hrs - @Risk	=RiskPearson5(6.1405,30474,RiskShift(-1622.1),RiskName("SE Hrs"))
PM Hrs - @Risk	=RiskLoglogistic(23.126,585,1.9424,RiskName("PM Hrs"))

To provide consistent correlated results from dataset to simulation, it is recommended to calculate the correlation coefficient based on Spearman's method

Consistency in Application of Correlation

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- **Example of consistent application of correlation in Simulation Model**

CORRELATE SEED RVs

	Seed X	Seed Y
Uniform Rand()	0.965	0.574
Converted to Std Norm	1.816	0.186
Corr Matrix	X	Y
	1	
	0.5	1
Cholesky	1	0
	0.5	0.866
Transpose Cholesky	1	0.5
	0	0.866
Multiply by Std Norm	X	Y
	1.816	1.069
Convert back to Uniform	X	Y
	0.965	0.857

CONDUCT TRIALS

	Seed X	Seed Y	Result X	Result Y
Trial 1	0.625	0.836	3.318	4.469
Trial 2	0.947	0.844	4.612	4.515
Trial 3	0.010	0.025	0.671	0.072
Trial 4	0.718	0.470	3.576	2.887
Trial 5	0.368	0.286	2.664	2.152
:	:	:	:	:
Trial 45	0.176	0.401	2.070	2.623
Trial 46	0.486	0.541	2.965	3.154

Convert Seeds into Normal Distribution outputs:

- Results X = Normal(3,1)
- Results Y = Normal(3,1.5)

CORRELATE RESULTS

Correlation (X -Y)	Seeds	Results
Pearson	0.545	0.597
Spearman	0.519	0.519

Calculated correlation of Seeds and Results after 46 trials:

- Spearman correlation is consistent from seed to result
- Pearson correlation is not consistent from seed to result

Results

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- **Spearman Rank Correlation Coefficient:**
  - Used in most simulation packages such as @RISK and Crystal Ball
  - Remains consistent from seeds to results
  - Does not rely on linear relationships
- **Pearson**
  - Is not used in simulation packages, however is found organically in Excel ("Correl" Function)
  - Does not remain consistent from seed to results
  - Statistically derived, but relies solely on presence of linear relationships

When injecting historical correlation into a simulation, it is best to calculate the Spearman rank correlation coefficient as it is consistent from application to results and guarantees the outputs will match the correlated inputs

Spearman User-Defined Function for Excel

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Steps for Inserting Spearman User-Defined Function (UDF) into Excel

- *Step 1:* Open Excel File where Spearman will be used
- *Step 2:* Hold "Alt" and Press "F11" to open VBA editor
- *Step 3:* Navigate to Insert > Module
- *Step 4:* Paste code from this slide into new Module
- *Step 5:* Use function in Excel by typing:
  - =Spearman(*Range1*, *Range2*) and pressing Enter

Key Assumptions:

1. Data is arranged in columns
2. There are no duplicate values within a Range

```
Public Function Spearman(Arr1 As Range, Arr2 As Range)

  Dim xiArr() As Integer
  Dim yiArr() As Integer
  Dim SumdiSq
  Dim Rows As Integer

  Rows = Arr1.Rows.Count

  If Arr2.Rows.Count <> Rows Then
    MsgBox "Ranges do not have the same number of rows. Try again."
    Exit Function
  End If

  ReDim xiArr(1 To Rows)
  ReDim yiArr(1 To Rows)

  SumdiSq = 0

  For i = 1 To Rows
    xiArr(i) = Application.WorksheetFunction.Rank(Arr1(i, 1), Arr1)
    yiArr(i) = Application.WorksheetFunction.Rank(Arr2(i, 1), Arr2)
    SumdiSq = SumdiSq + (xiArr(i) - yiArr(i)) ^ 2
  Next

  Spearman = 1 - ((6 * SumdiSq) / (Rows * (Rows ^ 2 - 1)))

End Function
```

Dividing by Average Rates

The Problem (1 of 2)

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- **In order to estimate the cost of a SW intensive program, most analysts use some form of the following equation:**
  - SLOC X Productivity Rate X Hourly Rate = $; Where
    - SLOC is some form of code count or distribution;
    - Productivity Rate is viewed as either SLOC/HR or HR/SLOC; and
    - Hourly Rate is viewed as $/HR
  - Leaving 2 main options for the calculation:

$$\frac{SLOC}{\left(\frac{SLOC}{HR}\right)} \times \frac{\$}{HR} = \$ \; or \; SLOC \times \frac{HR}{SLOC} \times \frac{\$}{HR} = \$$$

  - Either equation will work *iff* SLOC/HR is a single data point used as an analogy and therefore:

$$\frac{1}{\frac{SLOC}{HR}} = \frac{HR}{SLOC}$$

  - This is not necessarily the case, however, when using a dataset to determine an ***Average Productivity***

The Problem (2 of 2)

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- **When using an average productivity some estimates erroneously come forward with the following productivity assumption in their equation:**

$$\frac{1}{\frac{\sum_{i=1}^{n}\frac{SLOC}{Hr}}{n}} = \frac{\sum_{i=1}^{n}\frac{Hr}{SLOC}}{n}$$

- **In reality, when broken down further we see that:**

$$\frac{1}{\frac{\sum_{i=1}^{n}\frac{SLOC}{HR}}{n}} = \frac{n}{\sum_{i=1}^{n}\frac{SLOC}{HR}} \neq \frac{\sum_{i=1}^{n}\frac{HR}{SLOC}}{n}$$

- **As a result, if Average Productivity is your metric of choice, use Hr/SLOC instead of SLOC/Hr:**

$$SLOC \times \frac{\sum_{i=1}^{n}\frac{HR}{SLOC}}{n} \times \frac{S}{HR} = S$$

- **Other alternatives to Average Productivity include using a Weighted Average or calculating the Geometric Mean for the dataset (instead of the Arithmetic Mean)**
  - Both the Wtd. Average and Geometric Mean will provide consistent inverse results for both SLOC/HR and HR/SLOC

Example

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1. DATASET

Contract	SLOC	HRS	SLOC/Hr	Hr/SLOC
Contract 1	2000	4000	0.50	2.00
Contract 2	3000	9000	0.33	3.00
Contract 3	5000	2500	2.00	0.50
Contract 4	10000	8500	1.18	0.85
Contract 5	12000	18000	0.67	1.50
Contract 6	4000	8800	0.45	2.20
Contract 7	6000	9600	0.63	1.60
Contract 8	9000	16200	0.56	1.80
Contract 9	9000	9000	1.00	1.00
Contract 10	8000	7600	1.05	0.95
Sum			8.36	15.40
n			10	10
Average			0.84	1.54

2. Myth

$$\frac{1}{\frac{\sum_{i=1}^{n}\frac{SLOC}{Hr}}{n}} = \frac{\sum_{i=1}^{n}\frac{Hr}{SLOC}}{n}$$

$$\frac{1}{\frac{8.4}{10}} = \frac{15.4}{10}$$

or

$$1.19 = 1.54$$

3. Reality

$$\frac{1}{\frac{\sum_{i=1}^{n}\frac{SLOC}{HR}}{n}} = \frac{n}{\sum_{i=1}^{n}\frac{SLOC}{HR}} = \frac{\sum_{i=1}^{n}\frac{HR}{SLOC}}{n}$$

$$\frac{1}{\frac{8.4}{10}} = \frac{10}{8.4} = \frac{15.4}{10}$$

$$\frac{1}{.84} = 1.19 = 1.54$$

When multiplying numbers together to develop a "factor," the inverse of the arithmetic average of a dataset will not equal the arithmetic average of the reciprocal dataset

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Dividing by a Distribution in a Simulation Model

The Problem

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- **Similar to "Dividing by an Average Rate," with other complications**

DATASET

Contract	SLOC	HRS	SLOC/Hr	Hr/SLOC
Contract 1	2000	4000	0.50	2.00
Contract 2	3000	9000	0.33	3.00
Contract 3	5000	2500	2.00	0.50
Contract 4	10000	8500	1.18	0.85
Contract 5	12000	18000	0.67	1.50
Contract 6	4000	8800	0.45	2.20
Contract 7	6000	9600	0.63	1.60
Contract 8	9000	16200	0.56	1.80
Contract 9	9000	9000	1.00	1.00
Contract 10	8000	7600	1.05	0.95

Different Methodology Options	Mean	St Dev
Option 1A: Lognormal (Sloc/Hr)	0.84	0.5
Option 1B: Lognormal (Hr/Sloc)	1.54	0.75
Option 2A: Lognormal Geo. (Sloc/Hr)	0.73	1.71
Option 2B: Lognormal Geo. (Hr/Sloc)	1.37	1.71
Option 3A:Lognormal Wtd Avg (Sloc/Hr)	0.73	0.46
Option 3B: Lognormal Wtd Avg (Hr/Sloc)	1.37	0.61

1. Parameters:
   1000 SLOC, $150/HR

2. Equation Options (w/ Productivity):

$$\frac{1000\ SLOC}{Lognormal(\mu,\sigma)\ of\ SLOC/HR} \times \frac{\$150}{HR} = \$$$

or

$$1000\ SLOC \times Lognormal(\mu,\sigma)\ of\ \frac{HR}{SLOC} \times \frac{\$150}{HR} = \$$$

3. Based on the equation choices above, and the proof in the previous section, there will be a problem with using the arithmetic mean for the lognormal equations for SLOC/HR and HR/SLOC
   - **1.54 is not the inverse of .84**

Simulation Results

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Simulation with lognormal distributions on productivity only

Only the Hr/Sloc Options have Mean Inputs and Mean Outputs match

Simulation Options	Mean Estimate Input				Mean Simulation Results		
Option 1A: Sloc/Hr	SLOC	Sloc/Hr	$/Hr	$	Mean $	St Dev	CoV
Lognormal(0.84, 0.50); CoV = 59%	1000	0.84	150	$ 179,336	$ 242,190	$ 143,174	59%
Option 1B: Hr/Sloc	SLOC	Hr/Sloc	$/Hr	$	Mean	St Dev	CoV
Lognormal(1.54, 0.75); CoV = 49%	1000	1.54	150	$ 231,000	$ 230,985	$ 112,217	49%
Option 2A: Geo. (Sloc/Hr)	SLOC	Sloc/Hr	$/Hr	$	Mean	St Dev	CoV
Lognormal(0.73, 1.71); CoV = 233%	1000	0.73	150	$ 205,262	$ 1,323,217	$3,026,591	229%
Option 2B: Geo. (Hr/Sloc)	SLOC	Hr/Sloc	$/Hr	$	Mean	St Dev	CoV
Lognormal(1.37, 1.71); CoV = 125%	1000	1.37	150	$ 205,262	$ 205,135	$ 252,729	123%
Option 3A: Wtd Avg (Sloc/Hr)	SLOC	Sloc/Hr	$/Hr	$	Mean	St Dev	CoV
Lognormal(0.73, 0.46); CoV = 63%	1000	0.73	150	$ 205,588	$ 286,544	$ 180,128	63%
Option 3B: Wtd Avg (Hr/Sloc)	SLOC	Hr/Sloc	$/Hr	$	Mean	St Dev	CoV
Lognormal(1.37, 0.61); CoV = 45%	1000	1.37	150	$ 205,588	$ 205,584	$ 91,599	45%

CoV's match in all Options

Regardless of Option, if you want the mean estimate and the mean of the simulation to be the same, productivity should be measured in HR/SLOC

Problem with Scaling

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- **Dividing by the min of a SLOC/HR distribution vs. multiplying by the max of an HR/SLOC distribution yields different results for a "max" output**

Example:

Option 2A: Geo. (Sloc/Hr)	SLOC	Sloc/Hr	$/Hr	$	Mean	St Dev	CoV
Lognormal(0.73, 1.71); CoV = 233%	1000	0.73	150	$ 205,262	$ 1,323,217	$3,026,591	229%

Min. of Lognormal (0.73,1.71) = .0014 ⇒ $\frac{1000\ SLOC}{.0014\ \frac{SLOC}{HR}} \times \frac{\$150}{HR} = \$107M$

Option 2B: Geo. (Hr/Sloc)	SLOC	Hr/Sloc	$/Hr	$	Mean	St Dev	CoV
Lognormal(1.37, 1.71); CoV = 125%	1000	1.37	150	$ 205,262	$ 205,135	$ 252,729	123%

Max. of Lognormal (1.73,1.71) = 33.15 ⇒ $1000\ SLOC \times 33.15\ \frac{HR}{SLOC} \times \frac{\$150}{HR} = \$5M$

Dividing by a distribution (particularly when close to zero) will yield inaccurate simulation results

Overall Conclusions

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- **Correlation**
  - *When injecting historical correlation into a simulation, it is best to calculate the Spearman rank correlation coefficient as it is consistent from application to results and guarantees the outputs will match the correlated inputs*
- **Dividing by Average Rates**
  - *When multiplying numbers together to develop a "factor," the inverse of the arithmetic average of a dataset will not equal the arithmetic average of the reciprocal dataset*
    - **Dividing by SLOC/HR is *not always* the same as multiplying by HR/SLOC**
    - **Using a weighted average or the geometric mean alleviates this problem**
- **Dividing by Distributions**
  - *Dividing by distributions can yield inaccurate results in simulations*
    - **Developing a distribution for productivity that is measured in HR/SLOC will yield a mean estimate with the same results as the mean of a simulation**

Common Thread: Reliance on a COTS simulation package should not be a substitute for thorough analysis and model verification/validation

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