Integrity - Service - Excellence

Including Escalation into CERs

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Outline

- Introduction
- Review of Nonlinear Regression
- Cost Improvement Curve Example
- Summary and Conclusions

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Introduction

- **Prior to developing a Cost Estimating Relationship (CER) the data are normalized**
  - Could be for scope or quantities
  - Usually includes some form of economic normalization
- **Currently a lot of debate about what are the "right" set of economic indices to use for normalization**
  - GDP deflator, PPI, ECI, CPI
- **Recent DoD Policy places responsibility on analyst**
  - AFI 65-502: "analysts should use information and methodologies that have the highest probability of accurately estimating the budget authority that will be required"

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Goal

- **Choice of specific inflation/escalation rate is generally subjective**
  - Could vary by analyst (for same data sets)
  - Could be based on local policy

- **Normalizing with inappropriate rates could create major program disconnects**
  - Rejecting a truly meaningful CER due to lack of fit
  - Accepting a misleading CER due to apparent fit

- **Rather than make a bad assumption about the appropriate inflation/escalation rates, infer it from the data!**
  - Add an escalation parameter to CER and compute the best fit parameter from the data
    - Airborne Radio Example (In paper)
    - Cost Improvement Curve Example
  - Analysis provides feedback about the quality of the computed parameters

Review of Nonlinear Regression

- **Consider a simple error function,** $e(x,\beta) \in \mathbb{R}^{\mathbf{1\times 1}} \to \mathbb{R}$
  - Assume convex with sufficient regularity
  - Expand around a point $e\left(x,\beta^{(k)}\right)$ using Taylor Series

$$e(x,\beta) \approx e\left(x,\beta^{(k)}\right) + \left[\frac{\partial}{\partial\beta} e(x,\beta)\right]_{\beta=\beta^{(k)}} \left(\beta - \beta^{(k)}\right)$$

  - If we assume that $e(x,\beta) = 0$ and rearrange terms

$$\beta = \beta^{(k)} - \left(\left[\frac{\partial}{\partial\beta} e(x,\beta)\right]_{\beta=\beta^{(k)}}\right)^{-1} e\left(x,\beta^{(k)}\right)$$

- **Given an estimate $\beta^{(k)}$, we can find new (better) estimate $\beta^{(k+1)}$**
  - As $k \to \infty$ then $e\left(x,\beta^{(k)}\right)$ approaches its minimum
  - The error function has be linearized about $\beta^{(k)}$

Review of Nonlinear Regression

- **For system of equations, we get**

$$\boldsymbol{D}^{(k)}\left(\boldsymbol{\beta}-\boldsymbol{\beta}^{(k)}\right)=-\boldsymbol{E}\left(\boldsymbol{x},\boldsymbol{\beta}^{(k)}\right)$$

where $\boldsymbol{D}^{(k)}=\left[\frac{\partial}{\partial\boldsymbol{\beta}}\boldsymbol{E}(\boldsymbol{x},\boldsymbol{\beta})\right]_{\boldsymbol{\beta}=\boldsymbol{\beta}^{(k)}}$ is the error system Jacobian at $\boldsymbol{\beta}^{(k)}$

  – Not square; Not invertible

- **Use Pseudoinverse**
  – Provides least squares solution
  – $\left[\boldsymbol{D}^{(k)}\right]^{+}=\left(\left[\boldsymbol{D}^{(k)}\right]^{T}\left[\boldsymbol{D}^{(k)}\right]\right)^{-1}\left[\boldsymbol{D}^{(k)}\right]^{T}$

- **Iterative Solution**

$$\boldsymbol{\beta}^{(k+1)}=\boldsymbol{\beta}^{(k)}-\left[\boldsymbol{D}^{(k)}\right]^{+}\boldsymbol{E}\left(\boldsymbol{x},\boldsymbol{\beta}^{(k)}\right)$$

Multiplicative Error

- **Given a CER with multiplicative error**
  - $y_i = f(x_i, \beta) \cdot (1 + \varepsilon_i)$

- **Error term equation is**
  - $\varepsilon_i(x_i, \beta) = \frac{y_i}{f(x_i,\beta)} - 1$

- **By the Chain Rule, the partial derivatives of the error function are given in terms of the CER partial derivatives**
  - $\frac{\partial}{\partial \beta} \varepsilon_i(x_i, \beta) = \frac{\partial}{\partial \beta}\left[\frac{y_i}{f(x_i,\beta)} - 1\right]$

$$= \frac{-y_i\left(\frac{\partial}{\partial \beta} f(x_i,\beta)\right)}{f(x_i,\beta)^2}$$

- **Other error term forms can similarly be written in terms of the CER partial derivatives**

Matrix Condition Number

- **Suppose**
  - $\boldsymbol{A}$ is a nonsingular matrix, $\boldsymbol{z}$ satisfies $\boldsymbol{Az} = \boldsymbol{b}$
  - $\tilde{\boldsymbol{z}}$ satisfies the perturbed problem $\widetilde{\boldsymbol{A}}\tilde{\boldsymbol{z}} = \widetilde{\boldsymbol{b}}$ with $\left\|\widetilde{\boldsymbol{A}} - \boldsymbol{A}\right\| \|\boldsymbol{A}\|^{-\mathbf{1}} < \mathbf{1}$

- **Then**

$$\frac{\|\boldsymbol{z} - \tilde{\boldsymbol{z}}\|}{\|\boldsymbol{z}\|} \le \frac{\kappa(\boldsymbol{A})}{\left(\mathbf{1} - \left\|\widetilde{\boldsymbol{A}} - \boldsymbol{A}\right\| \|\boldsymbol{A}\|^{-\mathbf{1}}\right)} \left(\frac{\left\|\boldsymbol{b} - \widetilde{\boldsymbol{b}}\right\|}{\|\boldsymbol{b}\|} + \frac{\left\|\boldsymbol{A} - \widetilde{\boldsymbol{A}}\right\|}{\|\boldsymbol{A}\|} \right)$$

  where $\kappa(\boldsymbol{A}) = \|\boldsymbol{A}\| \left\|\boldsymbol{A}^{-\mathbf{1}}\right\|$ is called the condition number

- **If A is singular, A^{-1} doesn't exist**
  - Define generalized condition number

$$\kappa(\boldsymbol{A}) = \|\boldsymbol{A}\| \|\boldsymbol{A}^{+}\| = \frac{\sigma_1}{\sigma_r}$$

    where σ_1 and σ_r are the largest and smallest singular values of $\mathbf{A}$
  - Singular values are the square roots of eigenvalues of $\boldsymbol{A}^T\boldsymbol{A}$

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Cost Improvement Curve Problem

- **Production Average Unit Cost (AUC) CER using learning and rate effect**
  - $AUC = \beta_0 \left(\frac{\left(L+\frac{1}{2}\right)^{\beta_1+1} - \left(F-\frac{1}{2}\right)^{\beta_1+1}}{\beta_1+1} \right) \cdot \left(L - (F-1)\right)^{\beta_2 - 1}$
  - $\beta_0 = T_1,\ \beta_1 = b,\ \beta_2 = c$

- **Add escalation term**
  - $AUC = \beta_0 \left(\frac{\left(L+\frac{1}{2}\right)^{\beta_1+1} - \left(F-\frac{1}{2}\right)^{\beta_1+1}}{\beta_1+1} \right) \cdot \left(L - (F-1)\right)^{\beta_2 - 1} \cdot (1+\beta_3)^{Yr}$

- **Potential Issues**
  - Production data may not be truly independent
  - Parameters can have similar effects
  - Certain production profiles could make solution difficult

Cost Improvement Curve Dataset

- **True parameters used to generate notional data**
  - Error terms were sampled from a normal distribution with $\mu = 0, \sigma = 0.05$

Parameter	True Value
β_0	100
β_1	-0.120
β_2	-0.201
β_3	0.018

Lot	F	L	Yr	True Cost	Err %	Noisy Cost
1	6	15	0	47.71	-8.6%	3.57
2	16	35	1	37.91	1.2%	14.19
3	36	85	2	28.95	2.8%	3.76
4	86	185	3	23.25	0.3%	16.68
5	186	385	4	12.79	7.2%	13.70
6	386	585	5	7.64	7.8%	8.23
7	586	785	6	12.11	0.7%	12.20
8	786	985	7	11.15	-9.0%	10.15
9	986	1185	8	13.58	-5.6%	12.82
10	1186	1385	9	26.91	7.0%	28.78

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Model Verification

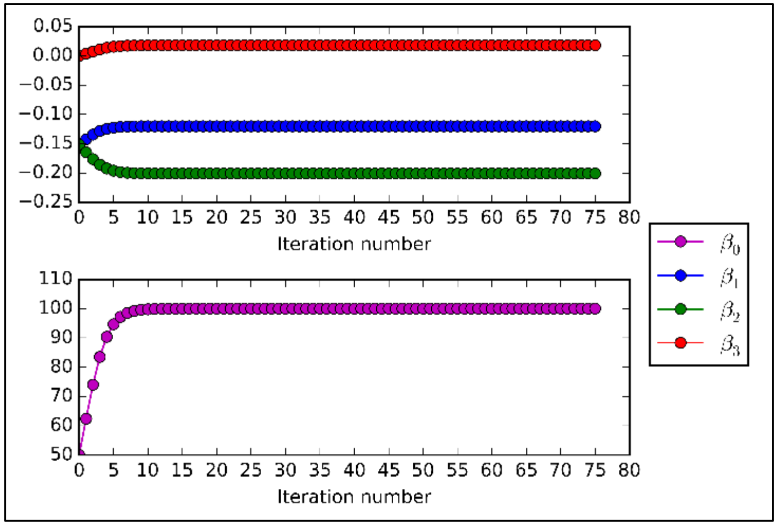
- **Check the model using the manufactured (perfect) data**

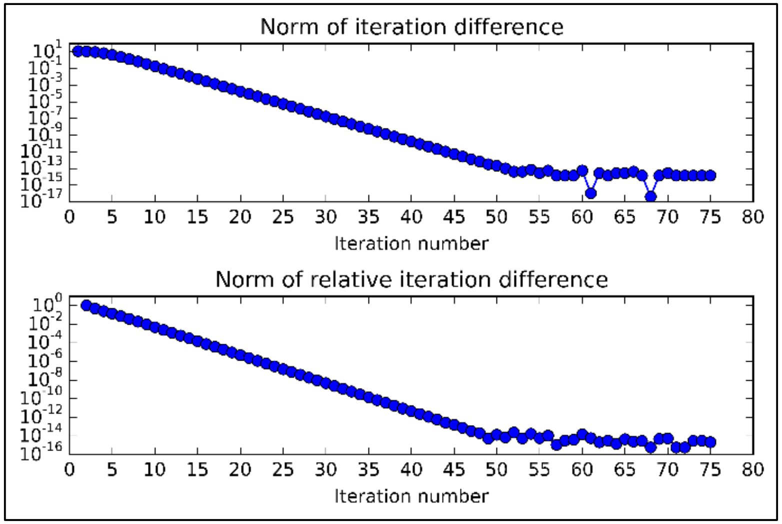
Parameter	Initial Value	Final Value	Final Error	Relative Error
β_0	50	100	-8.29e-09	8.29e-11
β_1	-0.152	-0.120	1.03e-10	8.59e-10
β_2	-0.152	-0.201	9.38e-11	4.67e-10
β_3	0	0.018	-1.48e-11	8.24e-10

- **Model computes the right values for the parameters**
  - Not quite exact though
  - Iterative methods have stopping criteria
  - Conditioning of problem may allow errors

Diagnostic Checks (Perfect Data)

- Check iteration values and differences

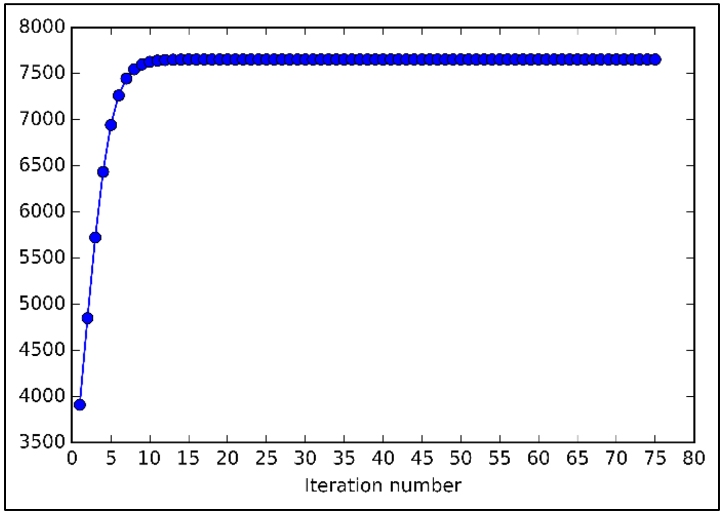




- Comments
  - After about 10 iterations the solution values don’t change significantly
  - After about 50 iterations the method has stalled

Diagnostic Checks (Perfect Data)

- **Check condition numbers and parameter correlations**



	β_0	β_1	β_2	β_3
β_0	1			
β_1	-0.717	1		
β_2	0.545	-0.973	1	
β_3	0.799	-0.978	0.911	1

- **Comments**
  - The condition number prevents a better solution
    - Initial guess yielded a better conditioned system Jacobian than the final solution
  - Strong correlation exists
    - Between the Rate and Learning curve parameters
    - Between the Escalation and both the learning and rate parameters
  - The model still converges to the right parameters (with perfect data)

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Model Application

- **Apply the model to the noisy cost data**

Parameter	Initial Value	Final Value	Final Error	Relative Error
β_0	50	95.047	4.954	5.0%
β_1	-0.152	-0.096	-0.025	20.6%
β_2	-0.152	-0.214	0.013	6.7%
β_3	0	0.014	0.004	24.3%

- **Maximum parameter relative error ~25%**
  - Parameter relative error about 5% overall
  - Noisy cost data relative error about 3.6% overall
- **Why ?**
  - Iterative Method Failure?
  - System conditioning?
  - Parameter correlation?

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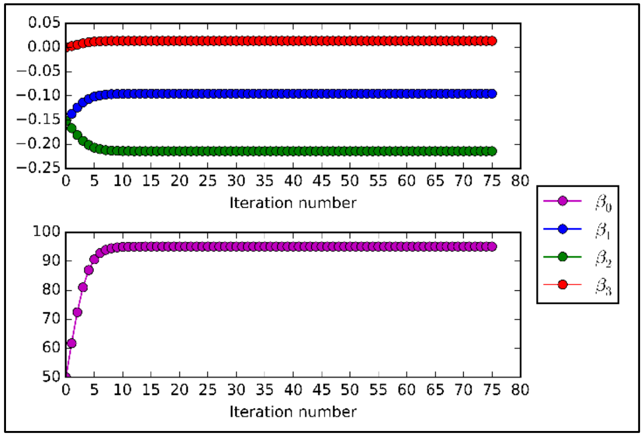
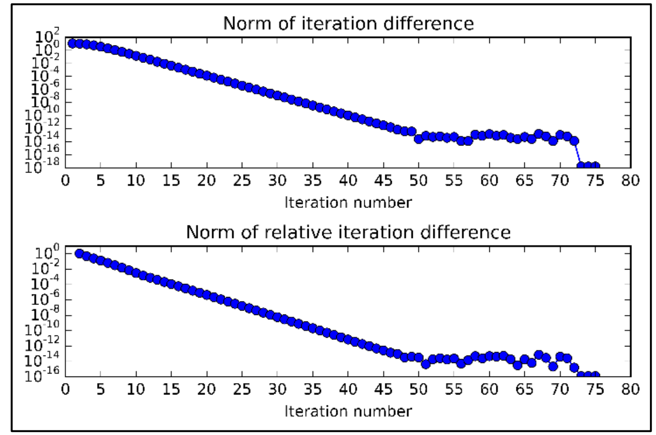
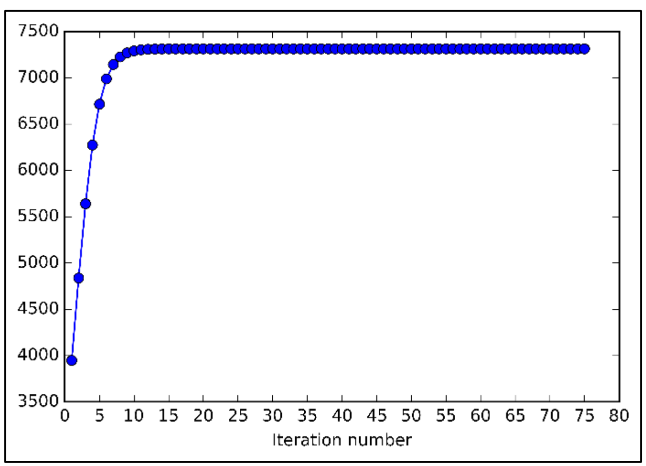
Diagnostic Checks (Noisy Data)

- **Check iteration values and differences**





- **Check condition numbers and parameter correlations**



	β_0	β_1	β_2	β_3
β_0	1			
β_1	-0.726	1		
β_2	0.558	-0.973	1	
β_3	0.807	-0.978	0.912	1

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Diagnostic Checks (Noisy Data)

- **The noisy cost data provides a slightly better conditioned system**
  - Reduces the spread of eigenvalues of $\left(\left[\boldsymbol{D}^{(k)}\right]^{T}\left[\boldsymbol{D}^{(k)}\right]\right)$
  - Does not reduce the correlation!

- **The condition number suggests the solution relative error could be magnified by more than 7000 times the relative data error**
  - Trivial examples exist that are worst case
  - In practice usually not that bad
  - For this problem 37% increase from the data relative error
  - Use specialized methods to mitigate the numerical errors
    - QR or SVD

- **The covariance matrix provides information useful for uncertainty quantification**

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Summary and Conclusions

- **The objective of this paper**
  - Highlight the critical normalization step
    - Choices are usually subjective, biased or at least require defending
  - Test under controlled conditions the feasibility of augmenting CERs with an escalation term
    - Condition numbers are a worst case, not a guarantee
    - Smarter input variable data sampling can reduce condition number
  - Provide a clear example for nonlinear regression

- **Future Work**
  - Apply method to identify a defendable, data driven commodity specific escalation rate

QUESTIONS?