A Mathematical Approach for Cost and Schedule Risk Attribution

Fred Y. Kuo
NASA Johnson Space Center

Abstract

The level of sophistication in cost estimate has progressed substantially in the last decade. This is especially true at NASA where, spearheaded by the Cost Analysis Division (CAD) of NASA Headquarters, cost and schedule risks in terms of confidence level have been codified. Cost estimate community now has a very good understanding on how to calculate portfolio risks, identify risk drivers and quantify their impacts, aided by various off-the-shelf simulation engines. However, one thing lacking in all these tools is the attribution of individual risks to the overall S-Curve. In this paper, the author attempts to delineate the concept of risk attribution in a portfolio of cost risks, develop mathematical formulation for this attribution. The author also attempts to extend the same formulation to the quantification of schedule risks by creating a portfolio of risks and tasks that affects the critical path. This approach has great potential in delineating and quantifying risk impacts on cost and schedule, and provides a metrics for trade studies in risk mitigation.

Background and Motivation

There are many off-the-shelf cost and schedule risk analysis tools on the market that allow user to perform cos/schedule risk analysis and generate S-Curves so that different confidence level can be estimated. One important aspect of risk analysis is to quantify the risk impact of each risk on the project. Tools like Primavera Risk Analysis allows user to perform sensitivity analysis with the aforementioned intent. However, the result is more confusing than illuminating. Figure 1 and Figure 2 serve to illustrate this point.

Figure 1 Cost Risk Sensitivity

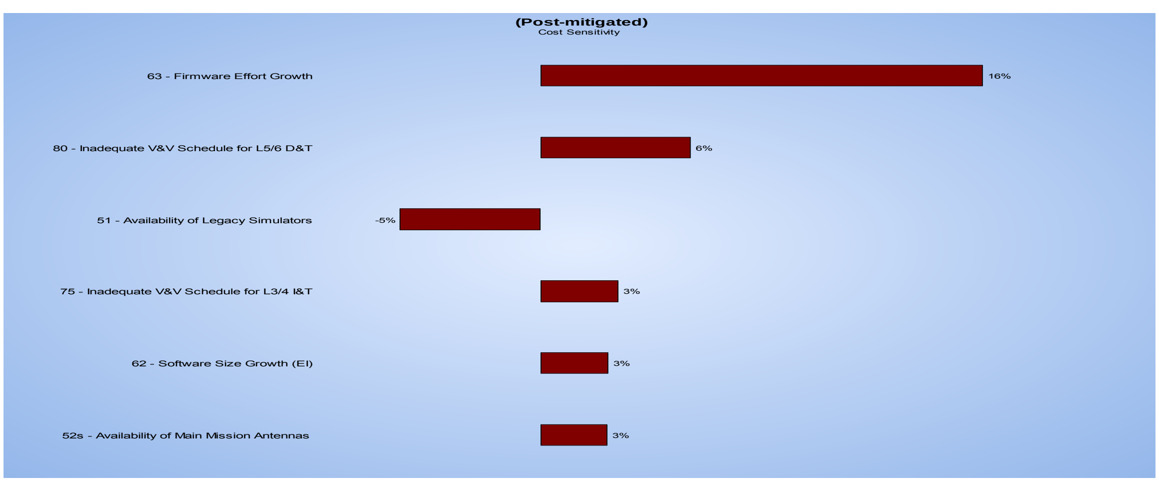
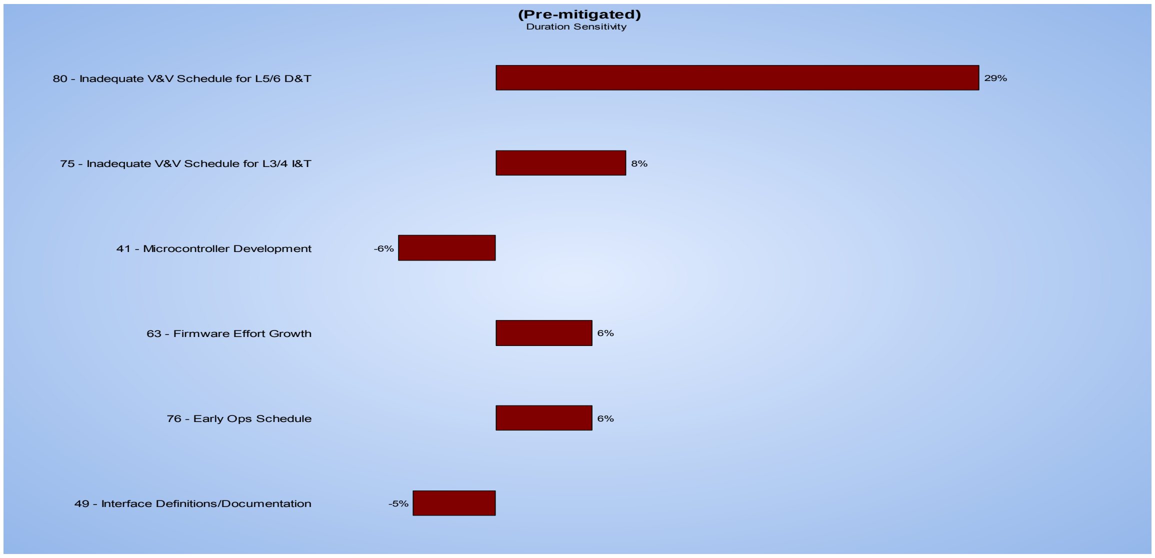


Figure 2 Schedule Risk Sensitivity



The definitions of cost and duration sensitivities are as follows:

1. The **cost sensitivity** of a risk event is a measure of the correlation between the occurrence of any of its impacts and the cost of the project (or a key task).

2. The **duration sensitivity** of a risk event is a measure of the correlation between the occurrence of any of its impacts and the duration (or dates) of the project (or a key task).

After working with this tool for many years and after a number of calls to the technical support, I am still befuddled by these two charts, especially the negative values. Somehow the risk of "availability of simulators and emulators" for testing is negatively correlated with project costs and schedule duration just does not sit well with common sense. The problem of using Correlation as a measure is that it is not a good risk metrics. There is little useful information that can be gleaned from these two charts.

A more useful and intuitively appealing approach is to calculate the contribution of each risk to a predefined risk measure so that the results are unambiguous. The advantages of this approach are manifold, not least of which is to provide a clear metrics for trade studies for a cost/benefit analysis for risk mitigation. Some of the current tool allows user to perform this manually by turning on/off each risk, rerun the simulation and records the difference. This approach becomes untenable if a large set of discrete risks has to be entertained. Therefore, it will be of value if a mathematical formulation can be developed and be easily integrated with existing cost estimate tools so that more insightful information regarding risk attribution can be obtained.

Defining Risk Measure

The financial industry went through a similar struggle on Wall Street, namely, how to capture the potential loss of a portfolio of assets with a certain probability. The Risk Metrics group of JP Morgan Bank came up with the concept of Value-at-Risk (VaR), and the concomitant methodology and process in calculating this number in the 1990s. Despite many criticisms, VaR is without question the most widely used and recognized risk metrics in the financial industry for risk management. The notable innovation of VaR is that it recognizes the portfolio volatility as a quantitative risk measure from which a portfolio loss both in terms of magnitude and probability can now be calculated.

This same idea can be extended to the cost and schedule risk analysis.

A project, with its constituents of subprojects or systems, can be considered as a portfolio. Therefore, the cost of a project is the statistical sum of its constituent parts. From cost estimate viewpoint, each item of the portfolio can be represented by a cost curve or its parametric equivalent of expected value (mean) and standard deviation. Borrowing from financial industry, the proper measure for risk is the concept of "volatility"; the dispersion around the mean or standard deviation. This risk measure is intuitively appealing as even cost estimate practitioners seem to have the same notion that the "steepness" of the cost curve is a measure of the "**risk**" of the estimate. It also turns out that by choosing portfolio standard deviation as the risk measure**, Euler's theorem on Homogeneous Function** provides a general method for additively decomposing this risk measure into its specific contributions. Appendix A describes the theorem.

Mathematical Formulation

The portfolio standard deviation can be represented as:

$$\sigma_p = \sqrt{w'\Sigma w}$$

Where

$$\Sigma = \begin{pmatrix} \sigma_{11} & \cdots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nn} \end{pmatrix}$$ is the covariance matrix

$w = [\, w_1, w_2, \dots, w_n]$ is a vector of portfolio weights

w' is the transpose of w.

The advantage of choosing σ_p as the risk measure is that now we can use Euler's theorem on Homogeneous Function of Degree One to decompose risks as:

$$\sigma_p = w_1 \frac{\partial \sigma_p}{\partial w_1} + w_2 \frac{\partial \sigma_p}{\partial w_2} + \ldots\ldots\ldots + w_n \frac{\partial \sigma_p}{\partial w_n}$$

Note that

$MCR_1 = \frac{\partial \sigma_p}{\partial w_1}$ is defined as the marginal contribution to risk measure by risk #1

Then

$CR_1 = w_1 * MCR_1$ is the contribution to risk measure by risk #1, and the total risk is the summation of each of the risk contribution CR_i

$\sigma_p = CR_1 + CR_2 + \ldots\ldots + CR_n$

So the percent contribution from each risk is:

$PCR_i = \frac{CR_i}{\sigma_p}$

Derivation of Marginal Contribution to Risk

The marginal contribution to risk (MCR) is just the partial derivative of σ_p with respect to weights. In matrix notation, the derivation is:

$$\frac{\partial \sigma_p}{\partial w} = \frac{\partial (w'\Sigma w)^{\frac{1}{2}}}{\partial w} = (w'\Sigma w)^{\frac{-1}{2}} (\Sigma w) = \frac{\Sigma w}{(w'\Sigma w)^{\frac{1}{2}}} = \frac{\Sigma w}{\sigma_p}$$

Where,

$\frac{\partial \sigma_p}{\partial w_i}$ is the i^{th} row of $\frac{\Sigma w}{\sigma_p}$

Example for a portfolio of 2 Risks

$$\sigma_p = \sqrt{w'\Sigma w}$$

$$\Sigma w = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_1\sigma_1^2 + w_2\sigma_{12} \\ w_2\sigma_2^2 + w_1\sigma_{12} \end{pmatrix}$$

$$\frac{\Sigma w}{\sigma_p} = \begin{pmatrix} \frac{w_1\sigma_1^2 + w_2\sigma_{12}}{\sigma_p} \\ \frac{w_2\sigma_2^2 + w_1\sigma_{12}}{\sigma_p} \end{pmatrix} = \begin{pmatrix} MCR_1 \\ MCR_2 \end{pmatrix}$$

$$CR_1 = w_1 MCR_1 \ ; PCR_1 = \frac{CR_1}{\sigma_p} = \frac{w_1^2\sigma_1^2 + w_1 w_2 \sigma_{12}}{\sigma_p}$$

$$CR_2 = w_2\, MCR_2 \ ;\ PCR_2 = \frac{CR_2}{\sigma_p} = \frac{w_2^2\sigma_2^2 + w_1 w_2 \sigma_{12}}{\sigma_p}$$

It is obvious that $\sum_{i=1}^{n} PCR_i = 1$

One of the advantages of this formulation is that it is not just limited to risks. The same formulation can be extended to include opportunities by merely reversing the signs of weights in the equation.

Numerical Examples

The following numerical examples serve to illustrate the validity of this formulation.

In this example, a portfolio of 2 cost risks is considered and each is a lognormal distribution. The properties of the distribution and the resulting calculations are shown in the following table.

	Mean	SD	Covariance	w(i)	MCR(i)	CR(i)	PCR(i)
Risk 1	10	0.2	0.0162	0.3338	0.1663	0.0555	0.3759
Risk 2	20	0.15	0.0162	0.6662	0.1383	0.0921	0.6240
Portfolio	30	0.15					

This result shows that Risk 1 contributes 37.6% of the overall variance, and Risk 2 contributes 62.4% of the overall variance.

In the next example, a portfolio of 5 risks with different risk characteristics were considered.

The following table shows the type of risks and their parametric values.

	Type	Mean	SD	W(i)	MCR(i)	CR(i)	PCR(i)
Risk 1	Lognormal	9.981	2.004	0.118	0.146	0.017	0.123
Risk 2	Lognormal	19.957	3.013	0.235	0.117	0.028	0.196
Risk 3	Triangular	18.312	4.236	0.216	0.189	0.041	0.292
Risk 4	Triangular	11.658	3.046	0.137	0.200	0.028	0.197
Risk 5	Normal	24.962	2.981	0.294	0.092	0.027	0.193
Portfolio		84.411	11.881	1.000		0.140	1.000

From this table, one can see that Risk 3 has the largest contribution to the variance even though it is the third from the weight stand point. The plots of risk contribution to portfolio mean and standard deviation are shown in Figure 3 and 4.

Figure 3

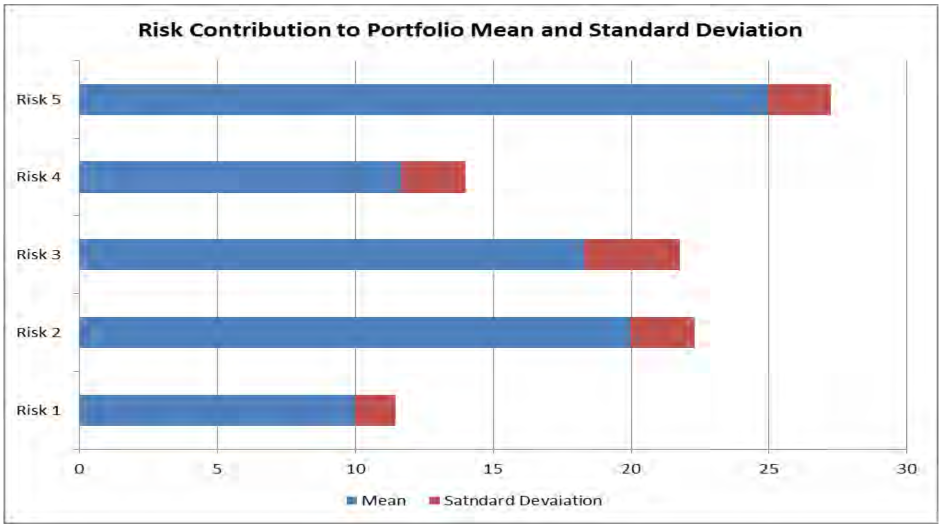
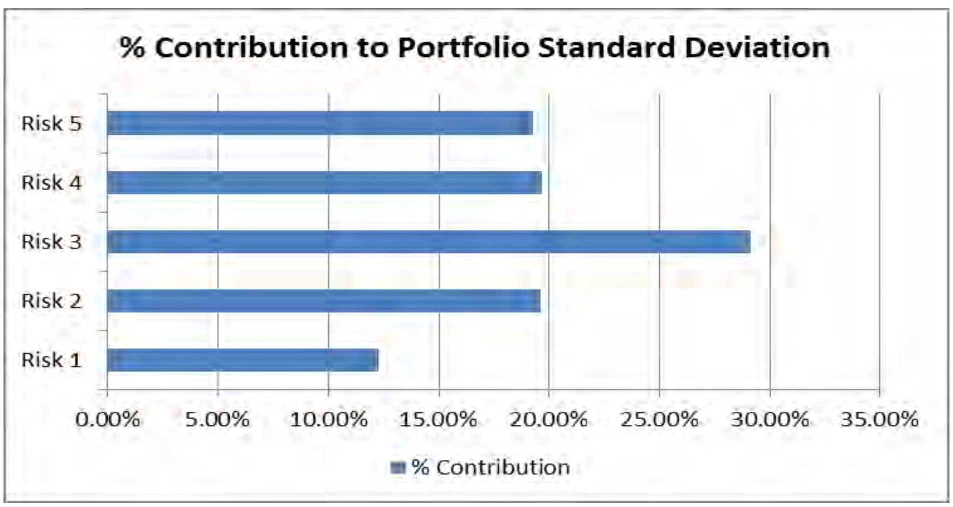


Figure 4



It should be emphasized that the sum of contribution from all risks is 100%. These results are more intuitively appealing because it is unambiguous and self-consistent.

The other advantage of this approach is that we can mix risks with "opportunities" by reversing the sign of weights as long as the sum of the weights is 1. This allows the "opportunity" to reduce the variance of the portfolio. As an example, if we now assume that Risk 1 in the previous example becomes an "opportunity". This means that there is an opportunity of savings with mean of 9.981 and standard deviation of 2.004. Therefore, this now reduces the mean to 64.908. The result of the rest of the calculation is shown in the following table.

	Type	Mean	SD	W(i)	MCR(i)	CR(i)	PCR(i)
Opp 1	Lognormal	-9.981	2.004	-0.154	0.099	-0.015	-0.106
Risk 2	Lognormal	19.957	3.013	0.307	0.116	0.036	0.246
Risk 3	Triangular	18.312	4.236	0.282	0.190	0.054	0.372
Risk 4	Triangular	11.658	3.046	0.180	0.198	0.036	0.247
Risk 5	Normal	24.962	2.981	0.385	0.091	0.035	0.242
Portfolio		64.908	9.376	1.000		0.144	1.000

The contribution to risks from the remaining 4 risks has now grown even though the overall risk has been reduced by the presence of the opportunity.

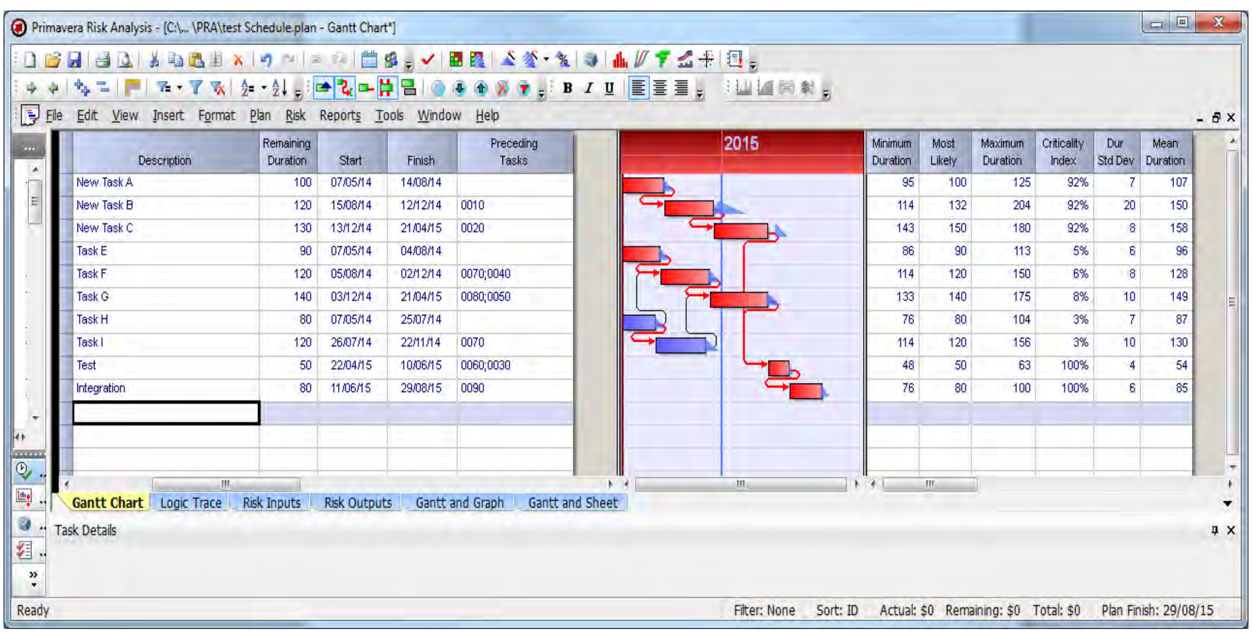
Schedule Risks

Unlike Cost risks, schedule risk is more complicated. The reason is that schedule is not fungible and one cannot statistically sum the mean and variance in the portfolio sense. However, we do know that only tasks and risks that are on the critical path will have an impact on the overall project duration. Therefore, we can proceed to treat tasks and risks that affect the critical path as a portfolio. We will examine this process in two stages.

Schedule Tasks Uncertainties

In order to test out this approach, a very simple schedule was created with only a few tasks but with two critical paths. Given the uncertainties of each task, the critical path will change throughout the simulation. Figure 5 shows this schedule in the PRA environment.

Figure 5



One advantage for using PRA is that it generates criticality index as a risk output. Since only tasks on critical path impact the overall project duration, one can surmise that if any task was on the critical path during any iteration would somehow contribute to the mean and standard deviation of the project duration. The following table shows the model outputs and compare to that of the calculation.

	L	ML	H	Cri_index	SD	Mean Duration	Mean* Cri_index	Sd* Cri_index
New Task A	95.00	100.00	125.00	94.10	6.93	106.67	100.38	6.52
New Task B	114.00	132.00	204.00	94.10	19.82	150.00	141.15	18.65
New Task C	143.00	150.00	180.00	94.10	8.40	157.66	148.36	7.90
Task E	86.00	90.00	113.00	3.90	6.32	96.33	3.76	0.25
Task F	114.00	120.00	150.00	4.20	8.25	128.00	5.38	0.35
Task G	133.00	140.00	175.00	6.40	9.56	149.33	9.56	0.61
Task H	76.00	80.00	104.00	2.60	6.55	86.67	2.25	0.17
Task I	114.00	120.00	156.00	2.20	9.64	130.00	2.86	0.21
Test	48.00	50.00	63.00	100.00	3.68	53.67	53.67	3.68
Integration	76.00	80.00	100.00	100.00	5.62	85.34	85.34	5.62
Portfolio					22.47	553.00	552.70	22.33
	Model output							
	Calculated							

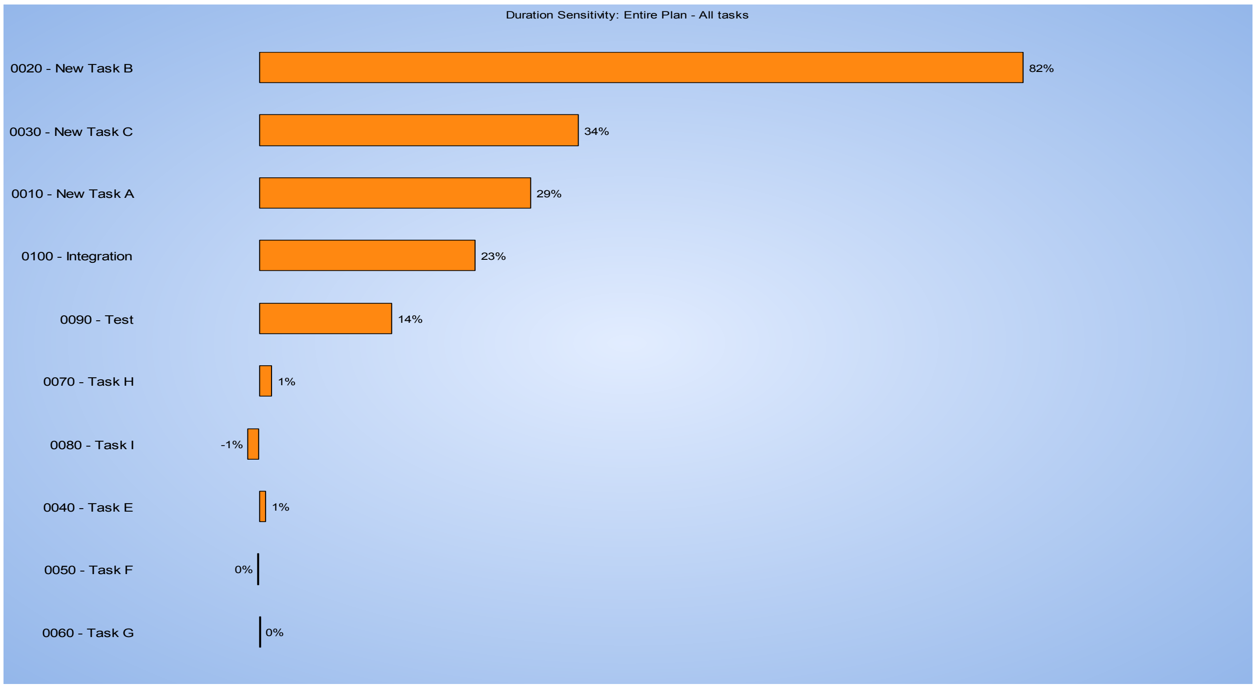
The result confirms that we can indeed treat the tasks as a portfolio if each is properly weighted by the criticality index.

Applying the same technique to this portfolio, the following results were obtained.

	Mean	SD	W(i)	MCR(i)	CR(i)	PCR(i)
New Task A	95.00	6.93	0.18	0.0634	0.0115	0.0478
New Task B	114.00	19.82	0.26	0.7299	0.1864	0.7733
New Task C	143.00	8.40	0.27	0.1336	0.0359	0.1488
Task E	86.00	6.32	0.01	0.0000	0.0000	0.0000
Task F	114.00	8.25	0.01	0.0000	0.0000	0.0000
Task G	133.00	9.56	0.02	0.0001	0.0000	0.0000
Task H	76.00	6.55	0.00	0.0000	0.0000	0.0000
Task I	114.00	9.64	0.01	0.0000	0.0000	0.0000
Test	48.00	3.68	0.10	0.0108	0.0010	0.0043
Integration	76.00	5.62	0.15	0.0401	0.0062	0.0257
Portfolio	553.00	22.47	1.0000		0.2410	0.9999

It is of interest to compare the output of “Duration Sensitivity” from PRA tool. This is shown in Figure 6

Figure 6



A more consistent and coherent graph can now be obtained, and they are shown in Figure 7 and 8

Figure 7

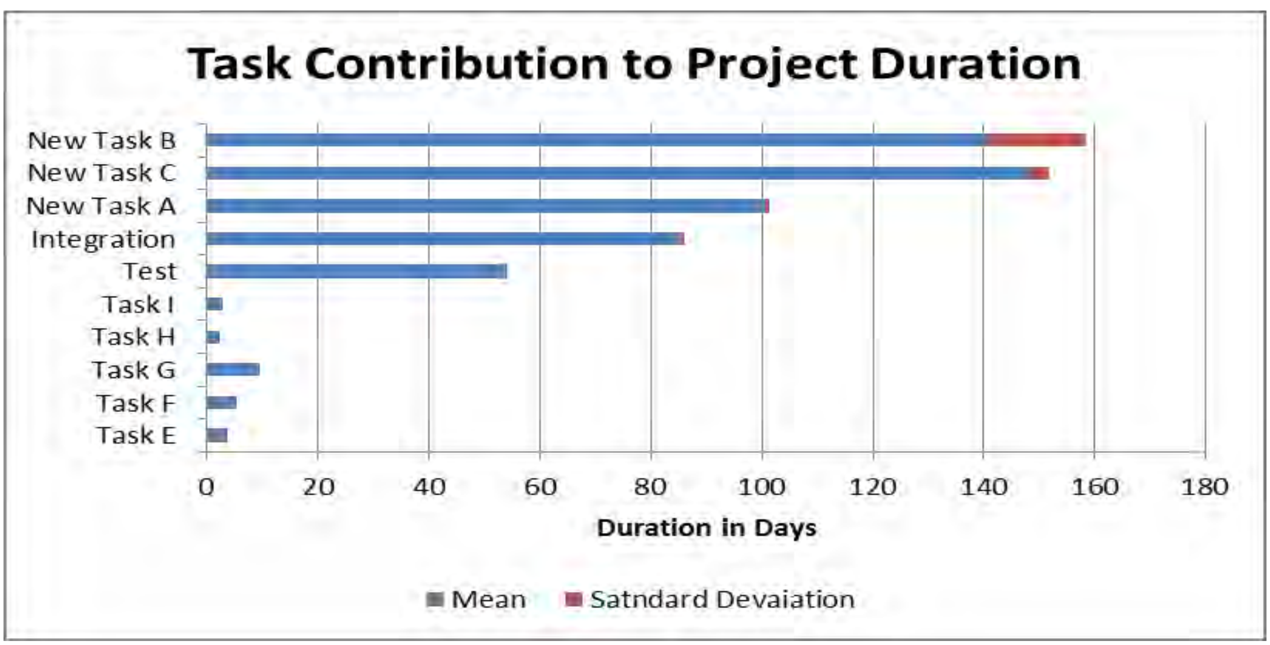
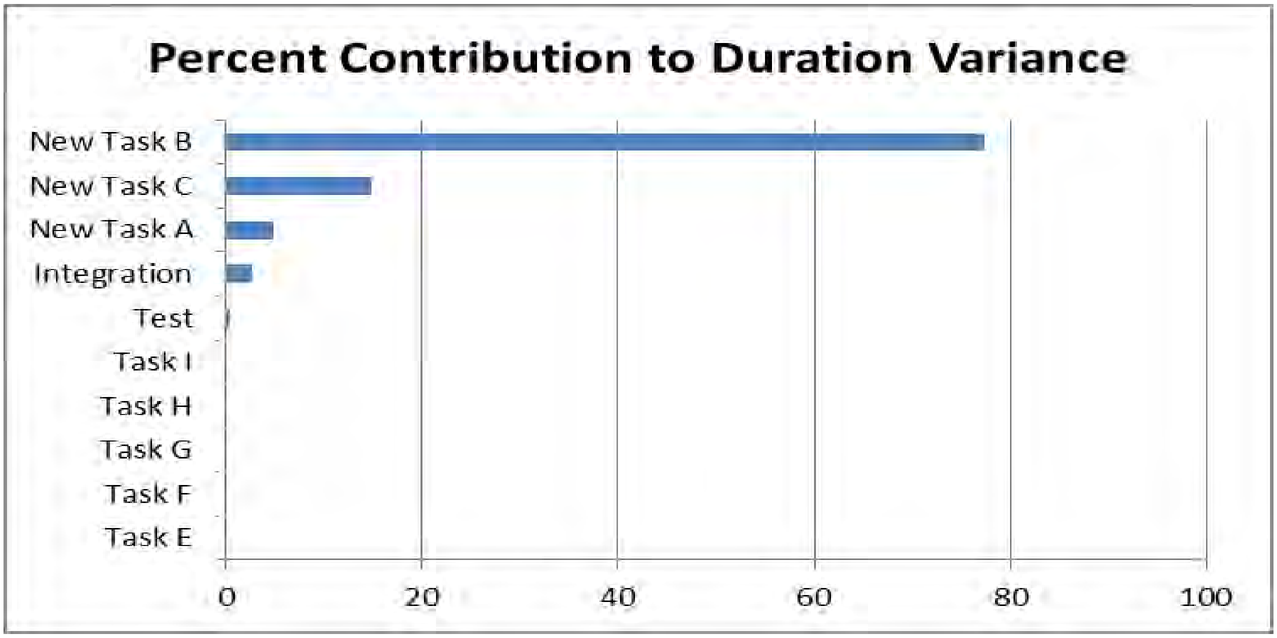


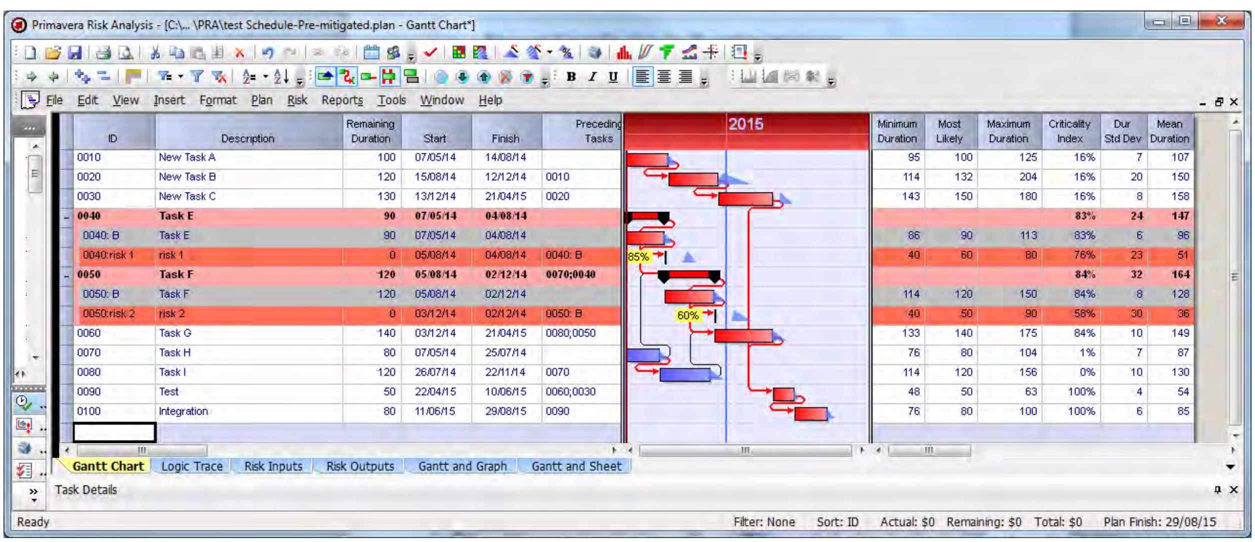
Figure 8



Schedule Task Uncertainties with Discrete Schedule Risks

The same procedure can be expanded to include schedule risks. Using the same schedule example with addition of 2 discrete risks, the new schedule model is shown in Figure 9.

Figure 9



There are two ways to handle the addition of discrete risks. One way is to treat the discrete risks as any tasks because all you need is the risk outputs like criticality index, mean and standard deviation which are internally generated through Monte Carlo simulation. In this case, the portfolio is expanded by two additional tasks. The other approach is to treat risks as an integral part of the tasks they are attached to. As shown in figure 9, impact of risk 1 and risk 2 were added to Task E and Task F respectively.

Once the risks are added to the schedule, it changes the dynamics of the critical path. The two risks have now pushed tasks F,E and G onto the critical path more often, and this results in their larger contribution to the project duration and standard deviation, as shown in Figure 10.

Figure 10

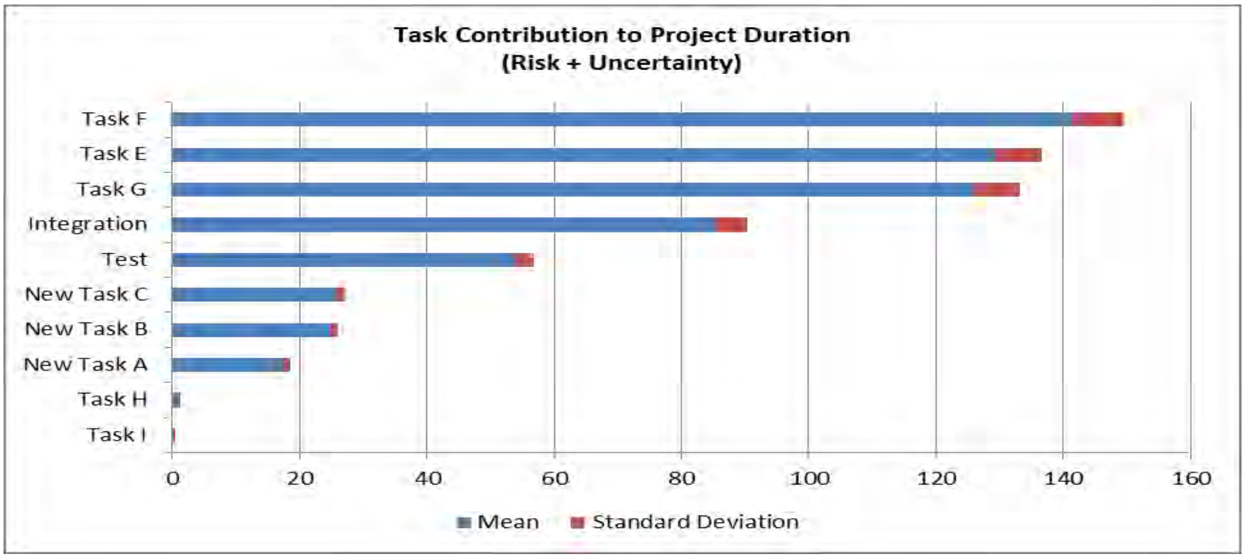
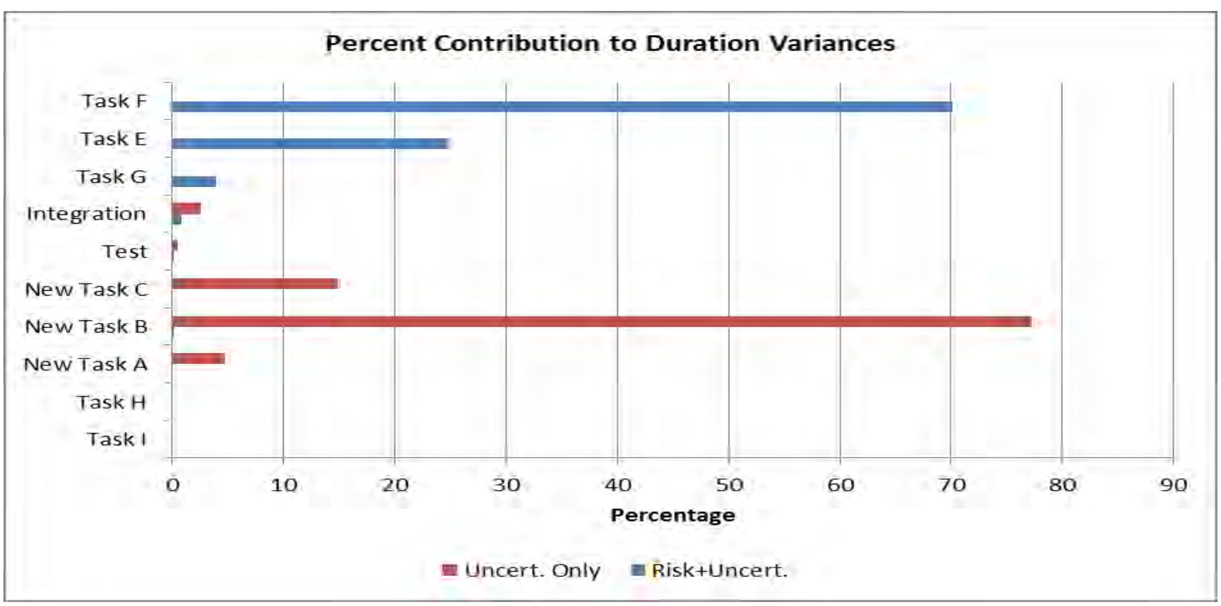


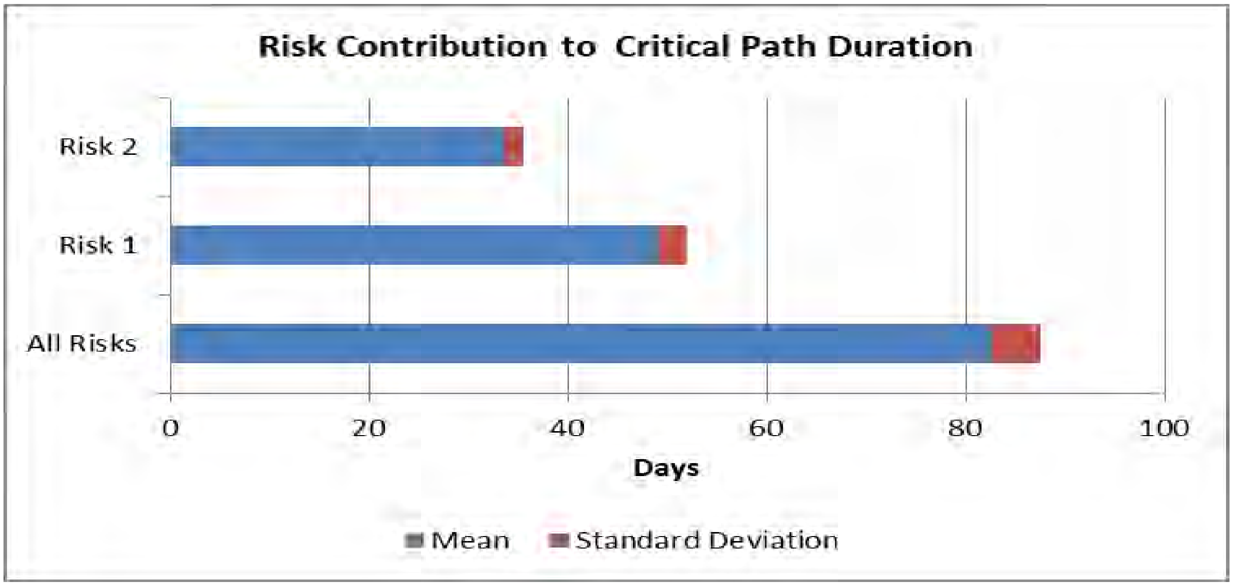
Figure 11 also serves to illustrate this point by comparing percent contribution to duration variance with and without discrete risks.

Figure 11



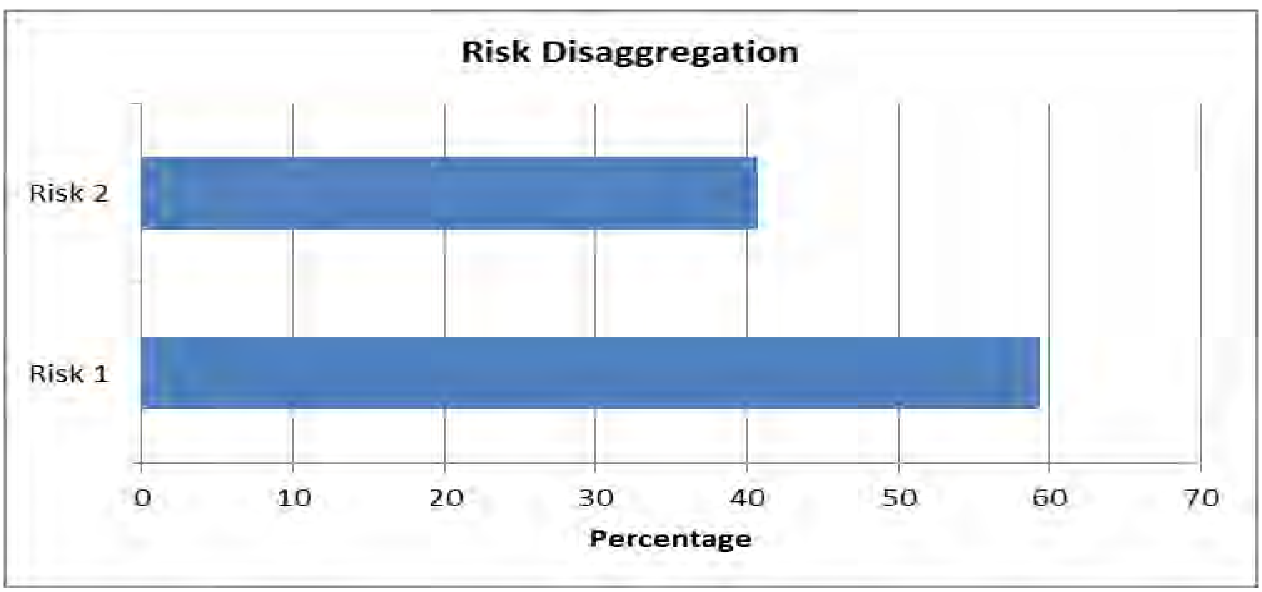
The contribution of discrete risks is shown in Figure 12.

Figure 12



The percentage contribution of each discrete risk can be disaggregated, as shown in Figure 13.

Figure 13



In general, once the algorithm is developed, there are many ways to present the data in a more meaningful and actionable ways. One interesting observation is the dynamic nature of schedule risk analysis because the addition of discrete risks can changes the critical path substantially.

Conclusion

The approach laid out here for the cost and schedule risk analysis is relatively easy to implement. It can be readily integrated into existing tools such as NAFCOM or Polaris. The reason one can decompose cost risks contribution is because portfolio standard deviation is a homogeneous function of degree one. Schedule risks, while far more complicate and dynamic than cost risks, can still be made to work by considering tasks on the critical path as a portfolio. It has been shown that, by using the criticality index, a portfolio of critical tasks can be assembled. This approach has great potential in delineating and quantifying risk impacts on cost and schedule, and provides a metrics for trade studies in risk mitigation.

Appendix A

Definition: Homogeneous function of Degree one

Euler's theorem states that if $f(x_1, x_2, \dots \dots .., x_n) = f(\boldsymbol{X})$ be a continuous, differentiable function of the variables $x_1, x_2, \dots \dots .., x_n$. Then *f is* homogeneous of degree ***k*** if for any constant *c*,

$f(c \cdot x_1, c \cdot x_2, \dots \dots .., c \cdot x_n) = c^k \cdot f(x_1, x_2, \dots \dots .., x_n)$, and that

$kf(\boldsymbol{X}) = \sum_{i=1}^{n} D_i f(\boldsymbol{X}) x_i$,

Where $D_i f(\boldsymbol{X})$ is the partial of $f(\boldsymbol{X})$ with respect to x_i.

In matrix notation we have$f(c \cdot \boldsymbol{X}) = c \cdot f(\boldsymbol{X})$.

The portfolio standard deviation , $\sigma_p = (\boldsymbol{X}'\boldsymbol{\Sigma}\boldsymbol{X})^{\frac{1}{2}}$ is such a function and can be proved in the following:

$\sigma_p(c \cdot \boldsymbol{X}) = ((c \cdot \boldsymbol{X})'\boldsymbol{\Sigma}(\mathrm{c} \cdot \boldsymbol{X}))^{\frac{1}{2}} = c \cdot (\boldsymbol{X}'\boldsymbol{\Sigma}\boldsymbol{X})^{\frac{1}{2}} = c \cdot \sigma_p(c \cdot \boldsymbol{X})$.

So using the above formulation, and setting k=1, we have:

$$f(\boldsymbol{X}) = x_1.\frac{\partial f(\boldsymbol{X})}{\partial x_1} + x_2.\frac{\partial f(\boldsymbol{X})}{\partial x_2} + \cdots .. + x_n.\frac{\partial f(\boldsymbol{X})}{\partial x_n} = \boldsymbol{X}'\frac{\partial f(\boldsymbol{X})}{\partial \boldsymbol{X}}$$

Where

X' is a row vector of *x*

$$\frac{\partial f(\boldsymbol{X})}{\partial \boldsymbol{X}} = \begin{pmatrix} \frac{\partial f(\boldsymbol{X})}{\partial x_1} \\ \vdots \\ \frac{\partial f(\boldsymbol{X})}{\partial x_n} \end{pmatrix}$$ is a nx1 vector

Now consider a portfolio of n risks $\boldsymbol{X} = (x_1, x_2, \dots \dots .., x_n)'$

E[**X**]= **μ,** and covariance(**X)** = **Σ**, then the portfolio mean and standard deviation is

$\mu_p = \boldsymbol{X}'\boldsymbol{\mu}$

$\sigma_p^2 = \boldsymbol{X}'\boldsymbol{\Sigma}\boldsymbol{X}, \quad \sigma_p = (\boldsymbol{X}'\boldsymbol{\Sigma}\boldsymbol{X})^{\frac{1}{2}}$

Both the portfolio mean and standard deviation are homogeneous functions of degree one in **X**

Appendix B Value-at-Risk Definition

Value-at-Risk has gained widespread use as a statistical risk measure. Value-at-Risk ("VaR") is defined as the maximum loss expected in a portfolio due to market fluctuations, in a given time period with a given probability. Alternatively stated, VaR is the amount of capital a firm needs in order to absorb portfolio losses with a stated probability of occurrence.
The mathematical definition of VaR is:

$$\text{Prob } [\Delta P(\Delta t, \Delta x) < -VAR] = 1 - \alpha$$

Where $\Delta P(\Delta t, \Delta x)$ is the change in portfolio value for changes in time Δt and change in asset price Δx and ($1-\alpha$) is the confidence level.

Thus, a one-day VaR of $1 million at the 95% ($\alpha = .05$) probability level implies that:

1. The probability of a one-day trading loss exceeding $1 million is 1%. A loss exceeding $1 million will be experienced 5 times every 100 days, on average.
2. The probability of a loss less than $1 million is 95%.

A first order analytical calculation for VaR is called Variance-Covariance VaR and is defined as follows:

Denote **ΔP** as the change in the value of the portfolio over time, that is, the difference between today's value and the value of the portfolio at the end of the time horizon. Assuming **ΔP** is normally distributed so the maximum loss that can occur within a 95% confidence interval is:

$ Exposure x 1.65 x σ - μ (1)

The exposure is equal to the market value of the portfolio. For very short-term horizons (one day), the mean of the distribution is often approximated by zero. The σ here is the portfolio standard deviation and can be calculated as $\sigma_p = (\boldsymbol{X}'\boldsymbol{\Sigma}\boldsymbol{X})^{\frac{1}{2}}$