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INFINITY SYSTEMS ENGINEERING

Meet the Overlapping Coefficient:

A Measure for Elevator Speeches

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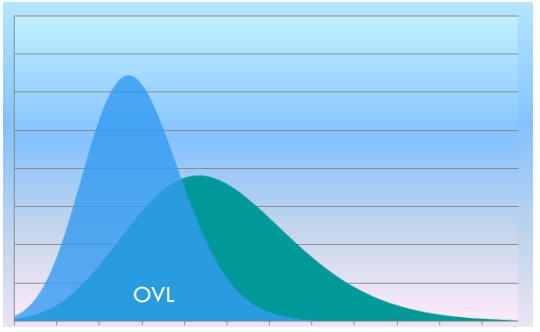
INFINITY SYSTEMS ENGINEERING

The Overlapping Coefficient

- What is it?
- Where did it come from?
- How might a cost analyst use it?
- How does one get the OVL?
- We want it now! I want it yesterday![1]

What is this coefficient?

- The overlapping coefficient (OVL) refers to the area under two probability density functions simultaneously.[2]



For continuous distributions:

$$OVL = \int_{R_n} \min[\, f_1(\mathbf{x}),\, f_2(\mathbf{x})]d\mathbf{x}$$

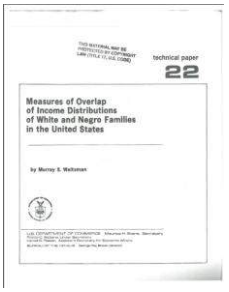
In discrete cases:

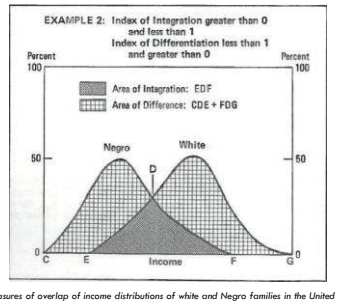
$$OVL = \sum_{\mathbf{x}} \min[\, f_1(\mathbf{x}),\, f_2(\mathbf{x})]$$

- The word "coefficient" means a measure of something
- Thus OVL is a measure of agreement or similarity[3]

Where did the OVL come from?

- In different form, OVL dates to the early days of Karl Pearson, ~ 1895
- Reportedly, explicit use begins in 1970[3] by economist Murray Weitzman to compare income distributions[4]

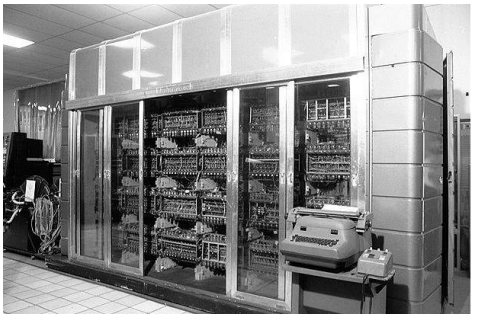




- Graphics from: Weitzman, M. S. (1970). *Measures of overlap of income distributions of white and Negro families in the United States*. Washington: U.S. Bureau of the Census; [for sale by the Supt. of Docs., U.S. Govt. Print. Off.

Where did the OVL come from?

- Biostatisticians at UAB Huntsville develop & define OVL as currently used[3,6,7,8] ~ 1980's -1990's
- However. . . story is much richer – Guess who's involved?
- Here's a clue:
- Johnniac?



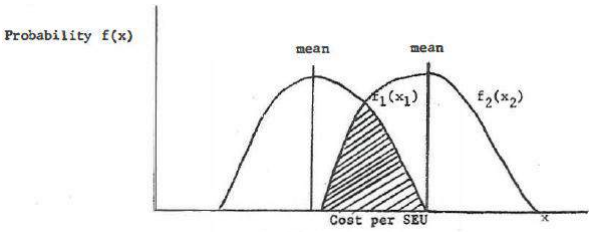
http://ed-thelen.org/comp-hist/Shustek/ShustekTour-02.html

Where did the OVL come from?

- Yep. . . RAND Corporation!
- Modern, explicit use of the OVL in the continuous case may be found earlier – at the birthplace of Weapon Systems Analysis & Cost Analysis
- 1958 - Ed Berman, RAND consultant & Harvard trained economist uses overlapping distributions to compare weapon system alternatives[9]

Where did the OVL come from?

- Here's the evidence. . .



Overlapping Cost-Probability Functions

- Here's Dr Berman's calculus. . .

$$V_2 = \int_{x_1=0}^{\infty} \int_{x_2=0}^{x_1} (x_1 - x_2)\, f_2(x_2)\, dx_2\, f_1(x_1)\, dx_1$$

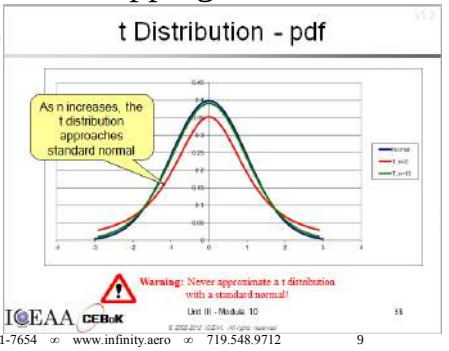
- Graphics from: Berman, E. B. (1958). Toward a new weapon system analysis. Santa Monica, Calif: Rand Corp.

[OBTW. . . historical context] Where did the OVL come from?

- Berman's paper, written for David Novick[10] (the "father of cost analysis[11]"), is an earlier use of probability theory to model cost uncertainty than is commonly known
  - Berman modeled conceptually and at the total system cost level
- Appears to be lay groundwork for later developments in cost uncertainty analysis
  - Method of Moments – Steven Sobel, MITRE, 1965[12]
  - Monte Carlo simulation – Paul F. Dienemann, RAND 1966[13]
- Dr Paul Garvey credits Sobel for pioneering the method of moments technique to create a probability distribution of total system cost[14]
- Sobel worked for Berman at MITRE[15]
- . . . and Sobel cites Berman's work in his 1965 paper!

How might a cost analyst use it?

- What's the OVL good for?
- Comparing theoretical weapon system models, etc.
- Also good for comparing probability models of different form - note these 3 overlapping distributions
- OVL ~ .86 for N(0,1), t(2)
- Models share 86% area
- Illustrates convergence of t to normal distribution

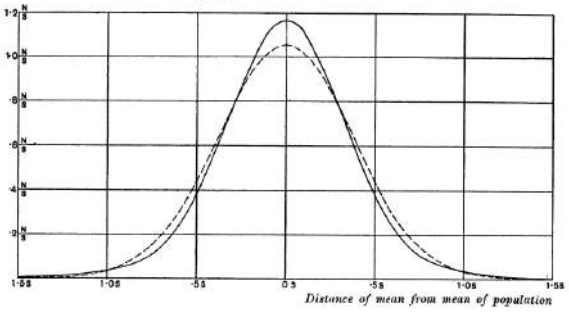


More context. . .

- Look familiar? - Ur case of previous graphic
- Would you believe that simulation was used?

DIAGRAM II. Solid curve $y=\frac{N}{\Omega}\times\frac{8}{7}\cdot\frac{6}{5}\cdot\frac{4}{3}\cdot\frac{2}{\pi}\cos^{10}\theta, \quad x/s=\tan\theta.$

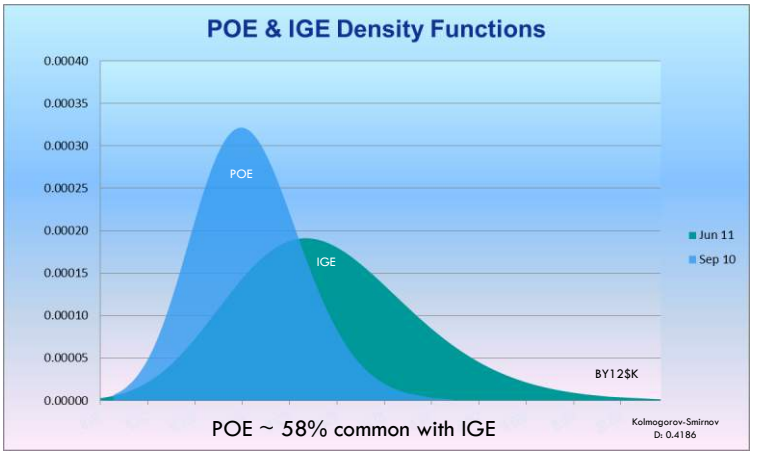
Broken line curve $y=\frac{\sqrt{7}\cdot N}{\sqrt{2\pi}\cdot s}e^{-\frac{7x^2}{2s^2}}$, the normal curve with the same S.D.



- Student (1908a). *The probable error of a mean.* Biometrika VI, 1-25.

How might a cost analyst use it?

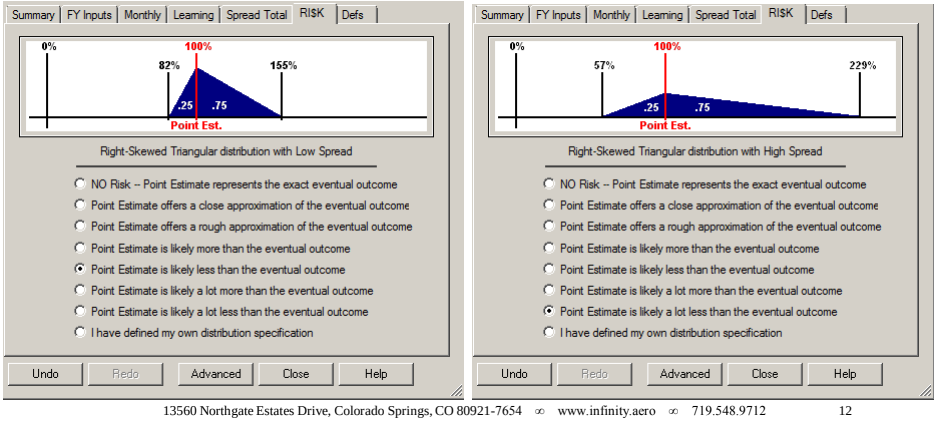
- Summarize change between estimates



How might a cost analyst use it?

- Find degree of similarity between input risk shapes

$$\overline{OVL} \sim .60$$

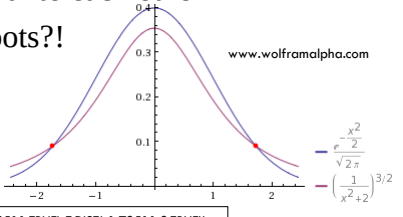


How does one get the OVL?

- Compute area using intersecting points of overlapping distributions
  - Most distributions will intersect 0, 1 or 2 times
- Normal versus t example
  - Intersections may be determined analytically or numerically
- Risk shape example
  - Intersecting points found visually
- In the case of data without known distributional form
  - More work is required. . .

How does one get the OVL?

- For parameterized models, e.g., for N(0,1), t(2)
- Step 1: WolframAlpha
  - Set equations for densities equal to each other
  - Click enter. . . and complex roots?!
- Step 2: Excel
  - Plug the real roots into NORM.S.DIST & T.DIST:



=1-ABS(NORM.S.DIST(1.72511,TRUE)-T.DIST(1.72511,2,TRUE))-ABS(NORM.S.DIST(-1.72511,TRUE)-T.DIST(-1.72511,2,TRUE))

= 0.85786

Solutions:

$x \approx -0.606179\,i$

$x \approx 0.606179\,i$

$x_1 \approx$ -1.72511

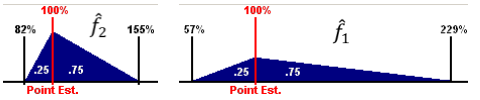
$x_2 \approx$ 1.72511

- Symbolically:

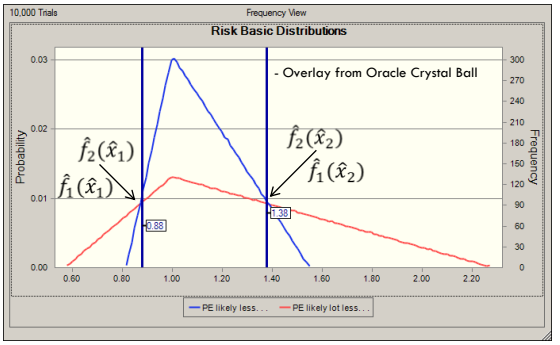
$$OVL = 1 - \left|\Phi(x_2) - F_2(x_2)\right| - \left|\Phi(x_1) - F_2(x_1)\right|$$

How does one get the OVL?

- For risk shape example



- Step 1: Overlay chart
  - Eyeball roots
- Step 2: Excel
  - Calculate
  - No triangular distribution function in Excel
  - See backup



$$OVL = 1 - \left|\hat{F}_1(1.38) - \hat{F}_2(1.38)\right| - \left|\hat{F}_1(.88) - \hat{F}_2(.88)\right|$$

Triangular CDF math for Excel

= IF(cost<low,0,IF(cost<mode,(cost-low)^2/((high-low)*(mode-low)),IF(cost<=high,1-(high-cost)^2/((high-mode)*(high-low)),1)))

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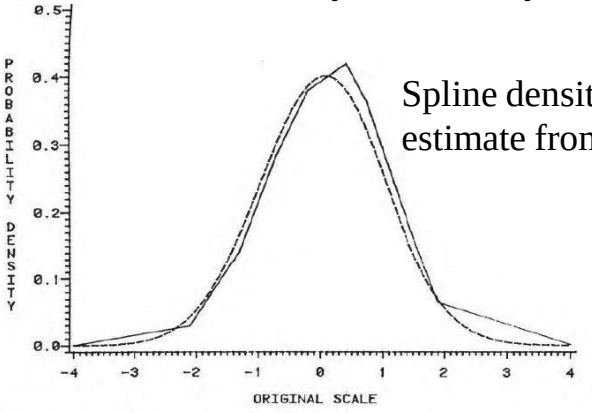
How does one get the $\widehat{OVL}$?

- Got data? – Then historically with density estimation



Spline density estimate from 1984

Figure 3.16 Spline density estimation: the spline-estimated density function obtained from a generated sample of 1000 standard-normal deviates. The estimated density is shown by the solid line, and the standard-normal density function is indicated by the broken line.

- Graphic From: Inman, H. F. (1984). *Behavior and properties of the overlapping coefficient as a measure of agreement between distributions*.

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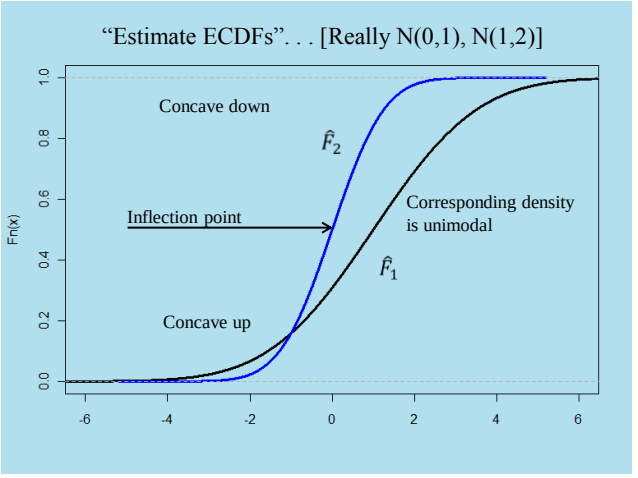
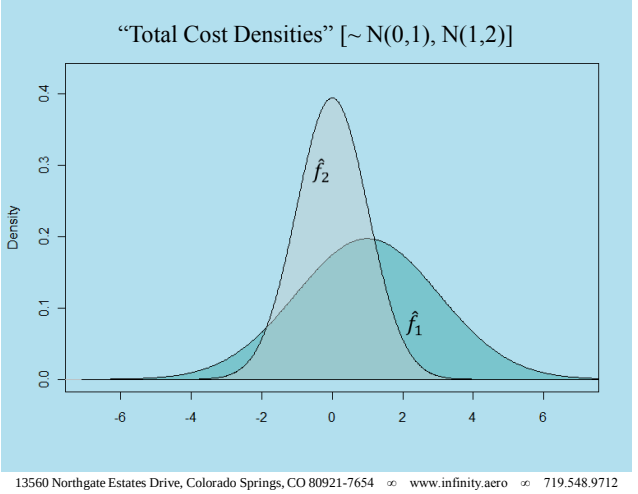
How does one get the $\widehat{OVL}$?

- From S-Curves! The story follows. .



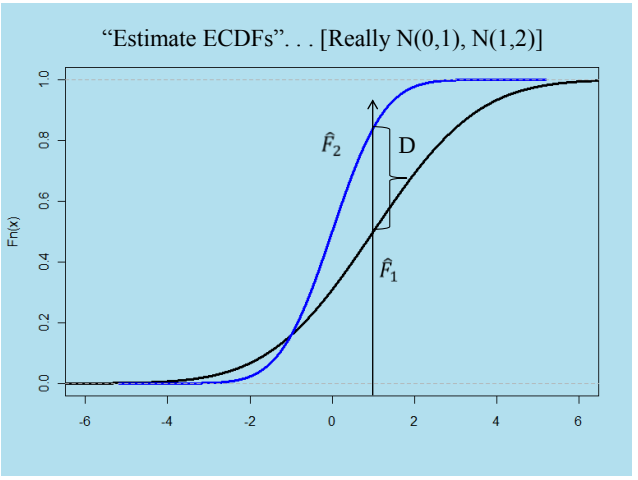
How does one get the $\widehat{OVL}$?

- On flip side of the fundamental theorem of calculus. . .



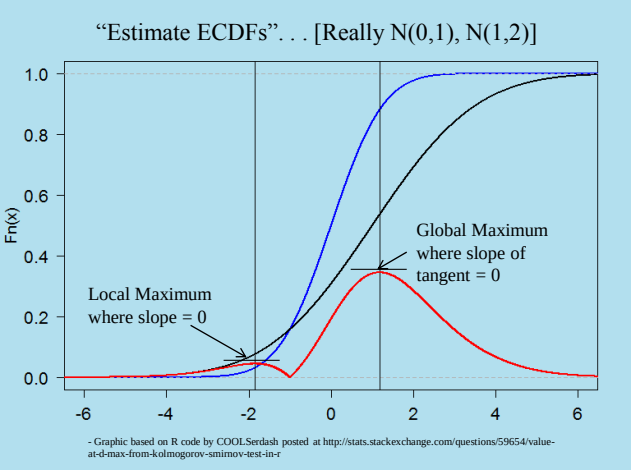
How does one get the $\widehat{OVL}$?

- Curves share a distance between them



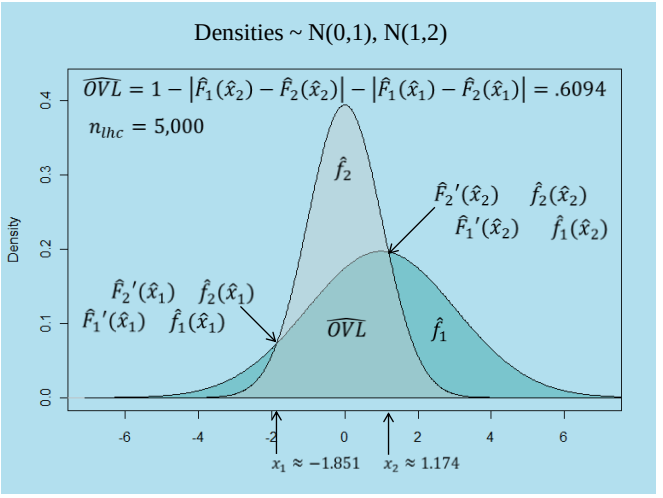
How does one get the $\widehat{OVL}$?

- Plotting every distance between S-Curves reveals. . .



How does one get away with this?

- Large sample size. . . 5,000 LHC trials



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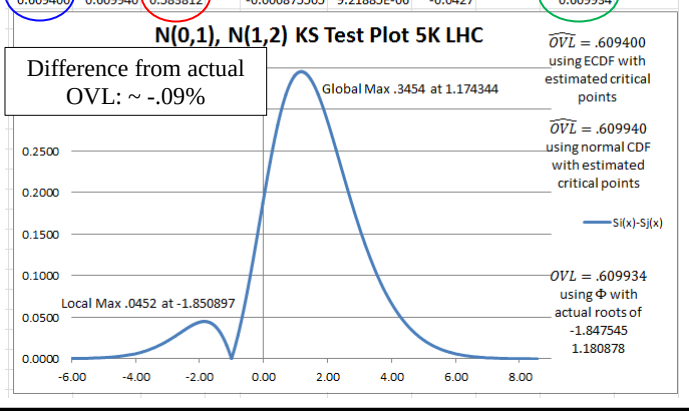
How good is this OVL?

Max	Crit Point	Crit Value	Row	Count Crit Vs	N(1,2)(x1)	N(0,1)x1	N(1,2)(x2)	N(0,1)(x2)
Global	1.174344	0.3454	7075	4	0.0772	0.0320	0.5346	0.8800
Local	-1.850897	0.0452	548	5				

OVLhat calculations			% Change from actual OVL			Actual OVL
ECDF	Excel CDF	Inman (spline)	ECDF	Excel CDF	Inman (spline)	Excel CDF
0.609400	0.609940	0.583812	-0.000875505	9.21885E-06	-0.0427	0.609934



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We want it now!

- "And if one has a method, its usefulness depends very much on whether it works quickly."
  - The Princeton Companion to Mathematics
- Free CD?
  - Includes Excel file showing how to calculate the KS Two Sample Test and generate an $\widehat{OVL}$ from the data

I want it yesterday!

- Special case for overlap with one intersection!
- Generate a couple hundred samples
- Paste into this web application and execute:
  - http://www.physics.csbsju.edu/stats/KS-test.n.plot_form.html

- Kirkman, T.W. (1996) Statistics to Use. http://www.physics.csbsju.edu/stats/ (May 15, 2014)

- Subtract the result from 1 and that's your $\widehat{OVL}$!
  - Remember the derivative of the max distance is where your probability density functions intersect
- More accuracy? Download the PAST tool for free
  - http://folk.uio.no/ohammer/past/

Hammer, Ø, Harper, D.A.T., Ryan, P.D. 2001. PAST: Paleontological Statistics software package for education and data analysis. Palaeontologia Electronica 4(1): 9 pp (May 15, 2014)

Overlap Wrap

- History of the common picture but obscure measure
  - Includes effort from the early days of cost analysis
- Application is wherever practical meaning is needed
  - In the context of comparing probability models or data
- Number quantifying OVL & $\widehat{OVL}$ is accessible
- Direct calculation from ECDF is the elegant method
  - But fitting distributions and using parameters could be quick
  - One intersection case yields quick answer with 1- D Statistic

References

Most citations pulled from WorldCat

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Backup

Contains:

- Excel for Risk Shape Example

Risk Shape Example

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	Calculations for OVLhat of ACE Risk Shapes													
2														
3														
4	Low and High parameters are multipliers about the mode from ACE help file													
5		Low	Mode	High										
6	f2	0.816	1.000	1.551	0	0	"PE likely less"							
7	f1	0.571	1.000	2.286	0	0	"PE likely a lot less"							
8														
9														
10	Visually determined intersections													
11	x1	0.88												
12	x2	1.38												
13														
14			f2(x1)	0.030287	=IF(B11<B6,0,IF(B11<C6,(B11-B6)^2/((D6-B6)*(C6-B6)),IF(B11<=D6,1-(D6-B11)^2/((D6-C6)*(D6-B6)),1)))									
15			f2(x2)	0.927797	=IF(B12<B6,0,IF(B12<C6,(B12-B6)^2/((D6-B6)*(C6-B6)),IF(B12<=D6,1-(D6-B12)^2/((D6-C6)*(D6-B6)),1)))									
16			f1(x1)	0.129776	=IF(B11<B7,0,IF(B11<C7,(B11-B7)^2/((D7-B7)*(C7-B7)),IF(B11<=D7,1-(D7-B11)^2/((D7-C7)*(D7-B7)),1)))									
17			f1(x2)	0.627821	=IF(B12<B7,0,IF(B12<C7,(B12-B7)^2/((D7-B7)*(C7-B7)),IF(B12<=D7,1-(D7-B12)^2/((D7-C7)*(D7-B7)),1)))									
18			OVLhat	0.60053										
19														
20			f2(x1)	0.030326	=CB.GetCertaintyFN(F6,B11)/100									
21			f2(x2)	0.927845	=CB.GetCertaintyFN(F6,B12)/100									
22			f1(x1)	0.129846	=CB.GetCertaintyFN(F7,B11)/100									
23			f1(x2)	0.627866	=CB.GetCertaintyFN(F7,B12)/100									
24			OVLhat	0.60050	=1-ABS(D21-D23)-ABS(D20-D22)									